BORDER RANKS OF MONOMIALS

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Abstract. Young flattenings, introduced by Landsberg and Ottaviani, give determinantal equations for secant varieties and provide lower bounds for border ranks of tensors. We find special monomial-optimal Young flattenings that provide the best possible lower bound for all monomials up to degree 6. For degree 7 and higher these flattenings no longer suffice for all monomials. To overcome this problem we introduce partial Young flattenings and use them to give a lower bound on the border rank of monomials which agrees with Landsberg and Teitler's upper bound.

1. Introduction

Given a homogeneous polynomial f of degree d in $n+1$ variables, what is the minimum number of linear forms ℓ_i needed to write f as a sum of d-th powers, $f = \sum_i \ell_i^d$? The answer to this question is the so-called Waring rank of f, denoted $\mathsf{Rank}\,(f)$. The (Waring) border rank is the answer in the limiting sense. That is, if there is a family $\{f_\epsilon \mid \epsilon > 0\}$ of polynomials with constant Waring rank r and $\lim_{\epsilon\to 0} f_\epsilon = f$, then we say that f has border rank at most r. The minimum such r is called the border rank of f, denoted $\mathsf{Brank}\,(f)$. These questions are field-dependent, but in this article we only work over an algebraically closed field of characteristic zero. Our main result determines the border ranks of all monomials:

Theorem 1.1. *Let* $\alpha = (\alpha_0 \geq \cdots \geq \alpha_n > 0) \in \mathbb{N}^{n+1}$.

$$\mathit{Brank}\,(x_0^{\alpha_0}\cdots x_n^{\alpha_n}) = \prod_{i=1}^{n}(1+\alpha_i).$$

Landsberg and Teitler [44, Theorem 11.2] proved the upper bound. We provide the unrestricted lower bound. One case of Theorem 1.1, when $\alpha_0 \geq \alpha_1 + \cdots + \alpha_n$, was also proved in [44, Theorem 11.3]. But to our knowledge, before this article even the border rank of $x\cdot y\cdot z\cdot w$ was unknown (Guan previously had the best lower bound of 7 [31]). Indeed, the search for lower bounds for ranks of tensors is quite elusive, see [22].

Carlini, Catalisano, and Geramita found the Waring rank of all monomials:

Theorem 1.2 ([15, 12])**.** *Let* $\alpha = (\alpha_0 \geq \cdots \geq \alpha_n > 0) \in \mathbb{N}^{n+1}$.

$$\mathit{Rank}\,(x_0^{\alpha_0}\cdots x_n^{\alpha_n}) = \prod_{i=0}^{n-1}(1+\alpha_i).$$

It is interesting to compare to other notions of rank [54, 5] and their ratios [35]:

Corollary 1.3. *For monomials the border rank, smoothable rank, and cactus rank all agree. The rank of a monomial x^α exceeds its border rank by a factor of $\frac{\alpha_0+1}{\alpha_n+1}$. In particular, $\mathit{Brank}\, x^\alpha = \mathit{Rank}\, x^\alpha$ if and only if $\alpha = (d,\ldots,d)$.*

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Our starting point is Landsberg and Ottaviani's Young flattenings [42]. Let V be a vector space with basis $\{x_0, \ldots, x_n\}$, $\varphi \in S^d V$ a homogeneous polynomial in the x_i, and $S_\lambda V$, $S_\mu V$ Schur modules with the property that $S_\lambda V \otimes S^d V$ contains $S_\mu V$ in its irreducible decomposition. One obtains a linear map, which depends linearly on $\varphi \in S^d V$,

$$\mathcal{F}_{\lambda,\mu}(\varphi) \colon S_\lambda V \to S_\mu V.$$

Linearity and subadditivity of matrix rank imply that if $m = \mathsf{Rank}\ \mathcal{F}_{\lambda,\mu}(x_0^d)$, and if φ has tensor rank r, then $\mathsf{Rank}\ \mathcal{F}_{\lambda,\mu}(\varphi) \leq mr$. The art in this construction is to find (λ, μ) such that m is small relative to $\min\{\dim S_\lambda V, \dim S_\mu V\}$, and such that one can demonstrate that $\mathsf{Rank}\ \mathcal{F}_{\lambda,\mu}(\varphi)$ is large. In our experience, the best pairs of partitions seem to be $(\lambda, (d, \lambda))$.

We show that $\text{Image}\left(\mathcal{F}_{\lambda,(d,\lambda)}(x_0^d)\right) = S_\lambda V_0$, where $V_0 = V/\langle x_0 \rangle$. In order to compute $\mathsf{Rank}\ \mathcal{F}_{\lambda,(d,\lambda)}(x^\alpha)$, we decompose the map into a direct sum using an action of the Lie algebra $\mathfrak{gl}(V)$. In particular, let X_i^j denote the basic element of $\mathfrak{gl}(V)$ sending x_i to x_j and acting by the induced action on Schur modules. Let $X_0^\nu := X_0^{\nu_1} \cdots X_0^{\nu_n}$ denote the monomial in the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(V))$. If λ is α-optimal (see Definition 3.7), then $\bigoplus_{\mu \preceq \alpha'} X_0^\mu S_\lambda V_0$ is a subvector space of $S_\lambda V$, and the Young flattening decomposes as

$$\mathcal{F}_\lambda(x^\alpha)(T) = \sum_{\mu+\nu=\alpha'} (-1)^{|\nu|} X^\nu(\mathcal{F}(x_0^d)(X^\mu.T)),$$

with target $\sum_{\nu \preceq \alpha'} X_0^\nu S_{d,\lambda} V_0 \otimes \langle x_0^d \rangle$. The summands in the target can fail to be full-dimensional or linearly independent, forcing the Young flattening to not produce the best possible bound on border rank (see Example 5.9). To get around this, we replace each summand with its unevaluated cousin $S_{d,\lambda} V_0 \otimes \langle x_0^d \rangle \otimes X_0^\nu$, which is full-dimensional. This yields a new map

$$\mathcal{F}_\lambda^p(x^\alpha)(T) = \sum_{\mu+\nu=\alpha'} (-1)^{|\nu|} (\mathcal{F}(x_0^d)(X^\mu.T)) \otimes X^\nu,$$

which we call a *partial Young flattening*. The following is our key technical result.

Theorem 1.4. *Let $V \cong \mathbb{C}^{n+1}$ and $V_0 = V/\langle x_0 \rangle$. Suppose $\alpha, \beta \in \mathbb{N}^{n+1}$ with $\alpha_0 \geq \cdots \geq \alpha_n$ and $\sum_i \alpha_i = \sum_i \beta_i = d$. Let $\lambda = (\sum_{i=1}^n \alpha_i, \sum_{i=1}^{n-1} \alpha_i, \ldots, \alpha_1)$, and let $\preceq$ denote dominance.*

(1) *If $(\beta_1, \beta_2, \ldots, \beta_n) \preceq (\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha'$ then the partial Young flattening*

$$\mathcal{F}^p_{\lambda,(d,\lambda)}(x_0^{\beta_0} \cdots x_n^{\beta_n}) \colon S_\lambda V \to \bigoplus_{\nu \preceq \alpha'} S_\lambda V_0 \otimes \langle x_0^d \rangle \otimes \langle X_0^\nu \rangle$$

has rank

$$\dim S_\lambda V_0 \cdot \prod_{i=1}^n (\beta_i + 1).$$

(2) *If $(\beta_1, \beta_2, \ldots, \beta_n) \preceq (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n) \preceq (\alpha_0, \alpha_n, \alpha_{n-1}, \ldots, \alpha_2)$, then the Young flattening*

$$\mathcal{F}_{\lambda,(d,\lambda)}(x_0^{\beta_0} \cdots x_n^{\beta_n}) \colon S_\lambda V \to S_{d,\lambda} V$$

has rank

$$\dim S_\lambda V_0 \cdot \prod_{i=1}^n (\beta_i + 1).$$

Remark 1.5. The extra hypothesis on β in Theorem 1.4(2) only matters when $d > 6$. The first case when $\alpha = \beta$ and this fails is when $d = 7$ and there is only one failure (see Example 5.9). When $d \geq 27$ it seems that more than half of the partitions fail the second condition. So it seems that Young flattenings of monomials rarely have maximal rank for large degree. However, many classes of monomials, like $(x_0 \cdots x_n)^k$, do have Young flattenings of maximal optimal rank.

Landsberg and Teitler gave the following upper bounds:

Theorem 1.6 ([44, Theorem 11.2])**.** *If $\alpha_0 \geq \cdots \geq a_n$, then* Brank $(x^\alpha) \leq \prod_{i=1}^n (\alpha_i + 1)$.

To obtain the lower bound, we combine the following ideas: the relation of ranks of (partial) Young flattenings to border rank in Propositions 3.4 and 4.4, and the ranks of partial Young flattenings computed in Lemma 3.5 (for x_0^d) and Theorem 1.4 (for x^α).

Theorem 1.7. *If $\alpha_0 \geq \cdots \geq \alpha_n$, then* Brank $(x^\alpha) \geq \prod_{i=1}^n (\alpha_i + 1)$.

Theorem 1.1 is an immediate corollary of Theorem 1.7 and Theorem 1.6. For Corollary 1.3 we note that our lower bound also agrees with Ranestad and Schreyer's upper bound for smoothable rank [54]. Because border rank is a lower bound for rank, we recover one case of Theorem 1.2, with proof independent of Ranestad and Schreyer's [54]:

Corollary 1.8. *The rank of $(x_0 \cdots x_n)^m$ is $(m+1)^n$.*

1.1. **Historical background and applications.** Homogeneous polynomials can be viewed as symmetric tensors, and computing ranks and border ranks of tensors is important for many applications, for instance in Algebraic Statistics [51] where the related notion is that of a mixture model of an independence model, or in Signal Processing [19] where one attempts to approximate an observed data tensor with one of lower rank, thus separating the observed data into signal and noise subspaces. Obtaining lower bounds for the rank and border rank of tensors is also important for Computational Complexity, because these bounds can be translated into bounds on algebraic complexity, [14, 40, 34, 43]. An interesting method used in this field is the notion of *shifted partial derivatives*, introduced by Kayal [38] to give the first example of an exponential lower bound for the rank of an explicit polynomial. See [24] and [26] for recent progress in this direction. For an introduction to Waring rank and related topics see the lecture notes [16] and the extensive references therein.

The question of generic Waring rank, that is the Waring rank for a Zariski dense open set of polynomials, was asked classically, and only solved in 1995 by the celebrated Alexander and Hirschowitz Theorem [1]. See [9, 52] for modern accounts and [32] for recent work on classifying secant defective varieties. In our setting, the Alexander and Hirschowitz Theorem says that the expected rank (based on the naive dimension count) is the correct one except for a small list of exceptions. So the set of polynomials for which we do not know the Waring rank has measure zero. On the other hand, it is still quite interesting to know the rank and border ranks of specific polynomials which might not be "generic", i.e. they lie in the measure zero set. Several algorithms for computing the decomposition of a given polynomial have recently been developed [6, 8, 47]. There is also interest in understanding the singularities of varieties of low-rank forms. The rank 2 case has been known classically. Han recently identified the singular locus in the rank 3 case, [33].

Landsberg and Teitler made a breakthrough on Waring rank [44]. Namely, they realized that the rank of a polynomial is intimately related its singularities. This revitalized

the study of Waring rank via apolar ideals, which had previously shown up classically in Sylvester's work and more recently in [18, 36]. For some examples of recent progress, see [13, 27]. Landsberg and Teitler also used symmetric flattenings to study the border rank of the permanent and determinant polynomials. Farnsworth recently obtained new lower bounds in the 3×3 case using Koszul flattenings and Young flattenings [25]. Guan recently obtained lower bounds for the product of variables (when d is odd) Koszul flattenings, [31], but his bounds are strictly smaller than the answer given in Theorem 1.1 as soon as $d = 4$. Ilten and Teitler recently computed the product ranks and tensor ranks of 3×3 permanent and determinants, [37]. Exciting progress on the complexity of the permanent versus the determinant was made recently by Landsberg and Ressayre, [39]. It will be interesting to see if the methods and results of this article can be pushed further to obtain lower bounds for border ranks of other polynomials like the permanent and determinant, like in [25, 31].

Another breakthrough by Buczyńska and Buczyński [11] introduced the notion of cactus rank, which is minimum degree of a zero dimensional scheme contained in the apolar ideal associated to the original polynomial. When the scheme is given by simple points the cactus rank agrees with the rank. See also [54, 7].

A highpoint in this line occurred when Catalisano, Geramita, and Gimigliano [15] proved Theorem 1.2, which solved the problem of the Waring Rank of monomials. Very shortly after Buczyńska, Buczyński, and Teitler gave another independent proof [12]. The basic idea of the proof of Theorem 1.2 is the so-called *apolarity lemma*, which relates rank to algebraic properties of the apolar ideal. Rank is bounded from below by bounding multiplicity of an ideal contained in the apolar ideal associated to the monomial, (see also [21]).

Geometrically, border rank is described via secant varieties. Let $V \cong \mathbb{C}^{n+1}$, and let S^dV denote the space of symmetric d-th order tensors on V, which is also isomorphic to the space of homogeneous polynomials of degree d on $n+1$ variables. Rank and border rank are unchanged by scaling globally by a non-zero scalar, so we will work with projective space.

The d-th Veronese variety is the image of the embedding

$$\begin{gathered} \mathbb{P}V \to \mathbb{P}(S^dV) \\ [v] \mapsto [v^d]. \end{gathered}$$

The r-th secant variety of the Veronese variety, denoted $\sigma_r(\nu_d\mathbb{P}V)$, is the Zariski closure of the points of Waring rank r, namely

$$\sigma_r(\nu_d\mathbb{P}V) = \overline{\{[f] \in \mathbb{P}S^dV \mid f = v_1^d + \cdots + v_r^d, \;\; v_i \in V\}}.$$

Since the Zariski and Euclidean topologies agree on constructible sets (such as the secant variety), $\sigma_r(\nu_d\mathbb{P}V)$ contains all points of border rank $\leq r$. Note that because Waring rank is not upper-semi-continuous, the rank can exceed the border rank.

There is much interest in finding equations for secant varieties, because, for instance, equations provide tests for membership and certificates for lower bounds on border rank. Ottaviani [49] constructed Aronhold's degree 4 invariant for plane cubics as a Pfaffian. In order to show that his construction produced a non-zero polynomial he demonstrated that the monomial $x_0x_1x_2$ produced a full rank matrix. Without commenting on this fact, Ottaviani proved that the border rank of $x_0x_1x_2$ is 4, which is equal to its rank. Theorem 1.4 generalizes of Ottaviani's work.

Landsberg and Ottaviani vastly generalized Ottaviani's construction to so-called Young flattenings, providing a large number of new classes of equations [42]. Their work provides a common construction for almost all known equations for secant varieties of classical varieties

such as Segre, Veronese and Grassmann varieties and their amalgamations. On the other hand, [4] and [20] describe two cases where special degree 6 polynomials occur as minimal generators of the ideal of certain secant varieties (found via Young symmetrizers) but no known Young flattening produces the equations. Young flattenings and partial Young flattenings, however, have the advantage over Young symmetrizers that they are determinantal, which can be easier to use when their matrices can be constructed.

One might wonder when the search for equations of secant varieties may stop. Sam recently demonstrated the existence of a bound on the degree of the minimal generators of the ideal of $\sigma_r(\nu_d\mathbb{P}V)$ that is independent of d and $\dim V$, [56]. The central idea, variants of which have been used by Aschenbrenner and Hillar [3], and Draisma and Kuttler [23], is to work with symmetric ideals in rings with infinitely many variables (See also [58, 57]). If one can show that the ideal is "Noetherian up to symmetry," this can provide a (non-constructive) guarantee that tensors of bounded rank are defined by equations in bounded degree not depending on the number of tensor factors. This method, however, does not typically give an explicit bound nor does it always give explicit equations.

Given a composition α of d, the general *Chow variety*, denoted $\mathrm{Chow}_\alpha \mathbb{P}V \subset \mathbb{P}S^dV$, consists of completely decomposable homogeneous polynomials of degree d with splitting type λ, i.e. all polynomials of the form $\ell_1^{\alpha_1}\cdots\ell_t^{\alpha_t}$ (up to scale), where ℓ_i are linear forms in V. $\mathrm{Chow}_d \mathbb{P}V$ coincides with the Veronese variety $\nu_d\mathbb{P}V$. Theorem 1.4 gives new equations for secant varieties of Veronese varieties and of general Chow varieties:

Theorem 1.9. *Let $V \cong \mathbb{C}^{n+1}$ and $V_0 = V/\langle x_0\rangle$. Let $m = \dim S_\lambda V_0$, $\alpha, \beta \in \mathbb{N}^{n+1}$ with $|\alpha| = |\beta| = d$ and let $r = \prod_{i=1}^n(\beta_i + 1)$.*

1. *$\mathrm{Chow}_\beta \mathbb{P}V \subset \sigma_r\nu_d\mathbb{P}V$, but $\mathrm{Chow}_\beta \mathbb{P}V \not\subset \sigma_{r+1}\nu_d\mathbb{P}V$.*
2. *Suppose $\lambda = (\sum_{i=1}^n \alpha_i, \sum_{i=1}^{n-1}\alpha_i, \ldots, \alpha_1)$. If β is such that $(\beta_1,\ldots,\beta_n) \preceq (\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $(\beta_1,\ldots,\beta_n) \preceq (\alpha_0,\alpha_n,\alpha_{n-1},\ldots,\alpha_2)$ in dominance order, then the $(rm+1)\times(rm+1)$ minors of the Young flattening $\mathcal{F}_{\lambda,(d,\lambda)}(\varphi)$ (for φ generic in S^dV) are non-trivial equations in the ideal of $\sigma_r\nu_d\mathbb{P}V$, and hence also non-trivial equations in the ideal of $\mathrm{Chow}_\beta \mathbb{P}V$.*

Lee and Sturmfels recently used projective duality to study the Euclidean distance degrees of coincident root loci [45], following classical work of Hilbert and more recent work [46]. Briand revitalized the study of Brill and Gordan's classical set-theoretic defining equations for the variety of completely reducible forms [10] and Chapalkatti addressed the problem for general coincident root loci [17]. Arrondo and Bernardi were able to compute some dimensions of secant varieties to Chow varieties, [2].

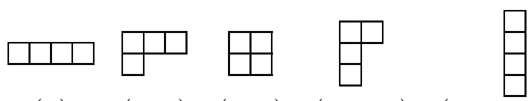
1.2. **Outline.** We review Young tableaux combinatorics and Representation Theory in Section 2, and we discuss crucial combinatorial ingredient for the proof of Theorem 1.4 we call *finding*. In Section 3 we carefully consider Landsberg and Ottaviani's construction of Young flattenings and their relation to bounds on border rank. In Section 4 we discuss further properties of Young flattenings, and their interaction with the Lie algebra $\mathfrak{gl}(V)$, showing how to construct explicit matrices representing these maps. In Section 4.3 we show our implementation of our equations in Macaulay2, utilizing the package PieriMaps, developed by Sam [55]. This leads to the introduction of partial Young flattenings. In Section 5 we describe a partial $\mathfrak{gl}(V_0)$ action Young flattenings, which facilitates several technical lemmas needed for our proof of Theorem 1.4.

2. Representation theory and Young tableaux

Standard references for this section are [29, 41]. The reader may wish to consult Ottaviani's lectures on projective invariants [50], or for an algorithmic point of view see [59, Ch. 4].

Throughout we let V denote an $(n+1)$-dimensional vector space over $\mathbb{C}$ and let $\mathrm{GL}(V)$ denote the invertible endomorphisms on V with maximal torus $\mathbb{T}(V)$. The dual vector space associated with V is denoted V^*. We often choose $\{x_0, \dots, x_n\}$ to represent a basis of V. With this choice of basis $\mathrm{GL}(V)$ is represented by $\mathrm{GL}(n+1)$, the invertible $(n+1)\times(n+1)$ matrices, and the torus $\mathbb{T}(V)$ is represented by the invertible diagonal matrices, denoted $\mathbb{T}^{n+1} \subset \mathrm{GL}(n+1)$.

2.1. **Young diagrams, tableaux and fillings.** A partition π of an integer d, denoted $\pi \vdash d$, is $\pi = (\pi_0, \dots, \pi_n)$ with $\pi_0 \geq \cdots \geq \pi_n$ and $\sum_{i=0}^{n} \pi_i = d$. To a partition $\pi \vdash d$ we associate a Young diagram, which is a box diagram with π_i boxes in row i, and sometimes denoted Y_π. For instance, the partitions of 4 and associated Young diagrams are



$$(4) \quad (3,1) \quad (2,2) \quad (2,1,1) \quad (1,1,1,1)\ .$$

In a partition, an exponent shall denote repetition. For example, $(1^4) = (1,1,1,1)$.

Associated with each partition λ we have the Schur module $S_\lambda V$, which is irreducible as a $\mathrm{GL}(V)$-module but not usually irreducible as a $\mathbb{T}(V)$ module.

2.2. **Fillings of tableaux and bases of Schur modules.** A filling of a Young diagram using the numbers $\{0, 1, \dots, n\}$ is an assignment of one number to each box, with repetitions allowed. A filled Young diagram is called a Young tableau. A standard filling is one in which the entries are strictly increasing in the both the rows and columns, while a semi-standard filling is one in which the entries are strictly increasing in the columns and weakly increasing in the rows. Standard tableaux and semi-standard tableaux are similarly defined. For a given partition π and alphabet Ω, respectively let $\mathrm{SYT}_\pi(\Omega)$ and $\mathrm{SSYT}_\pi(\Omega)$ denote the sets of standard and semi-standard tableau filled by letters from Ω.

Example 2.1. $\mathrm{SSYT}_{(2,1,0)}(\{0,1,2\}) = \left\{ \begin{array}{|c|c|}\hline 0&0\\\hline 1\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 0&0\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 0&1\\\hline 1\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 0&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 0&2\\\hline 1\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 0&2\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \right\}.$

2.3. **Dimensions of irreducible representations.** The irreducible representations of the symmetric and general linear groups are indexed by partitions. The standard tableaux index bases of the irreducible representations of the symmetric group. These representations are called Specht modules, denoted $[\pi]$. The semi-standard tableaux index bases of the irreducible representations of the general linear group. These representations are called Schur modules, denoted $S_\pi V$.

The numbers of standard and semi-standard tableaux, which respectively count the dimensions of the associated irreducible representations of the symmetric and general linear groups, are given by so called "hook-length formulas". To each box in a Young diagram we assign the hook length by counting that box together with the number of boxes directly to the right (the arm) and directly below (the leg). If the pair (i,j) denotes the location of a box in row i and column j in Y_π write $(i,j) \in Y_\pi$ and let $h_{i,j}$ denote the hook length of the hook cornered at box (i,j). It is convenient to record the hook lengths in a filling of the

tableaux. For example, the hook lengths of the Young diagram of shape $(4,3,2,1)$ are

$$\begin{array}{|c|c|c|c|}\hline 7&5&3&1\\ \hline 5&3&1\\ \cline{1-3} 3&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array}\,.$$

The number of semi-standard tableaux of shape $\pi \vdash d$ filled with numbers $\{0,1,\ldots,n\}$, and thus the dimension of the irreducible $\mathrm{GL}(n+1)$-module $S_\pi\mathbb{C}^{n+1}$, is given by

$$\dim S_\pi\mathbb{C}^{n+1} = \prod_{(i,j)\in Y_\pi} \frac{n+1+j-i}{h_{i,j}}. \tag{1}$$

It is convenient to write this dimension as a ratio of tableaux, taking the product of the contents of each of the tableaux. For instance,

$$\dim S_{3,2,1}\mathbb{C}^3 = \begin{array}{|c|c|c|}\hline 3&4&5\\ \hline 2&3\\ \cline{1-2} 1\\ \cline{1-1}\end{array} \Bigg/ \begin{array}{|c|c|c|}\hline 5&3&1\\ \hline 3&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array} = 8.$$

Above we enumerated the 8 semi-standard fillings of shape $(2,1,0)$ on alphabet $\{0,1,2\}$, which demonstrates that $\dim S_{2,1}\mathbb{C}^3 = 8$ as well. This is an instance of the following:

Lemma 2.2. *Suppose $V \cong \mathbb{C}^{n+1}$, $\pi \vdash d$ and $\pi = (\pi_0,\ldots,\pi_n)$. If $\pi_n > 0$ then the $\mathrm{GL}(V)$ modules $S_\pi V$ and $S_{\pi-(\pi_n^{n+1})}V$ are isomorphic. In particular, they differ by multiplication by a power of the determinant.*

2.4. **Weight Bases for Schur modules.** The standard action of $\mathbb{T}^{n+1}$ on V is by non-zero rescaling on each basis vector. The action on $V^{\otimes D}$ is the induced action. Specifically, the element $t = (t_0,\ldots,t_n) \in \mathbb{T}^{n+1}$ acts on the tensor monomial $x_0^{\otimes m_0} \otimes x_1^{\otimes m_1} \otimes \cdots \otimes x_n^{\otimes m_n}$ as multiplication by the scalar

$$t_0^{m_0} \cdots t_n^{m_n}.$$

the tuple $[m_0,\ldots,m_n]$ is the weight of the action, which clearly does not depend on the order of the terms in the tensor monomial. Note that weights are additive over tensor products.

Recall Schur-Weyl duality, which says that

$$V^{\otimes D} = \bigoplus_{\pi\vdash D} [\pi] \otimes S_\pi V,$$

as a $\mathfrak{S}_D \times \mathrm{GL}(V)$-module, where $[\pi]$ is a Specht module, and $S_\pi V$ is a Schur module, both associated to the partition π. We can identify an explicit instance of a representation $S_\pi V$ inside $V^{\otimes D}$ by fixing a particular element in $[\pi]$. Since $[\pi]$ has a basis indexed by standard tableaux filled with $\{1,\ldots,D\}$, we can pick the first tableau in the natural ordering: filling with columns filled one at a time from left to right then top to bottom. This element in $[\pi]$ fixes a Young symmetrizer $c_\pi \colon V^{\otimes D} \to V^{\otimes D}$ whose image is an explicit copy of $S_\pi V$ in $V^{\otimes D}$.

A basis of $V^{\otimes D}$ consists of tensor monomials. A weight basis of our copy of $S_\pi V$ is indexed by semi-standard fillings of Young diagrams of shape π and content from $\{0,\ldots,n\}$, denoted $\mathrm{SSYT}_\pi\{0,\ldots,n\}$. We treat the elements of $S_\pi V$ as formal linear combinations of filled tableau. The name *weight basis* is justified since Young symmetrizers are $\mathbb{T}^{n+1}$-equivariant maps and the tableaux represent images of monomials under the Young symmetrizer mapping. The weight of a tableau is determined by its content: $\omega(T)$ is the integer vector recording in its i-th coordinate the number of $\boxed{i}$'s occurring in T. The weight of a partition λ is the vector $\omega(\lambda) \in \mathbb{N}^{n+1}$, with $\omega(\lambda)_i$ equal to the number of columns of λ with height i, and is the weight of the highest filling of tableaux of shape λ.

Input: A tableau $T \in \mathrm{SSYT}_\lambda\{1,\dots,n\}$ with λ a α-optimal shape (Def. 3.7), and $\nu \preceq \alpha' = (\alpha_1,\dots,a_n)$.
Output: A generalized horizontal strip in T with content ν.

(1) Let β_i denote the number of $\boxed{i}$'s from the ith block of T.
(2) Choose ν_1 $\boxed{1}$'s from the first block of T.
(3) Repeat while $j \leq n$.
  (a) Choose the left-most $\min\{\nu_j, \beta_j\}$ $\boxed{j}$'s from the jth block of T.
    (i) If $\beta_j = \nu_j$ beak and increment j.
    (ii) If $\beta_j < \nu_j$ for each column in block j missing a $\boxed{j}$ choose the next smallest entry of that column, and let γ_i denote the number of $\boxed{i}$'s chosen in block j.
    (iii) Reuse the algorithm to find a generalized horizontal strip with content $(\nu_1 - \gamma_1, \dots, \nu_{j-1} - \gamma_{j-1}, \beta_j)$ in the first $j-1$ blocks.

FIGURE 1. Row finding algorithm (verified in the proof of Lemma 2.4).

2.5. **Shuffling rules.** A non-standard tableau can be written in the semi-standard basis by applying the shuffling rules, which can be found in many text books, but we prefer the formulation in [53, Lemma 3.16 and 4.18 (a,b)]:

Proposition 2.3. *The following relations hold among Young tableaux, (we suppress from the notation the parts of the tableau that don't change and only record the relevant subtableaux):*

(a) $\mathfrak{S}_n$ *acts by the sign representation on the columns. That is* $\begin{array}{|c|}\hline x\\\hline y\\\hline\end{array} = -\begin{array}{|c|}\hline y\\\hline x\\\hline\end{array}$, *in particular* $\begin{array}{|c|}\hline x\\\hline x\\\hline\end{array} = 0$.
(b) $\mathfrak{S}_n$ *acts by the trivial representation on columns of the same size. That is, if σ is a permutation of the entries of a tableau T that interchanges columns of the same size then $\sigma(T) = T$.*
(c) *Tableaux satisfy the truncated Plücker relation:* $\begin{array}{|c|c|}\hline x & z\\\hline y \\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline x & y\\\hline z \\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline z & x\\\hline y \\\cline{1-1}\end{array}$.

2.6. **Finding generalized horizontal strips with prescribed content.** A generalized horizontal strip, or GHS for short, is a sub-tableau of a given tableau which has no two boxes in the same column. The following lemma guarantees the existence of GHSs with certain content in semi-standard tableaux of a special shape. Later we'll see why these *monomial-optimal* shapes are important.

Lemma 2.4. *Let* $\alpha = (\alpha_0 \geq \cdots \geq \alpha_n)$, $\alpha' = (\alpha_1,\dots,\alpha_n)$, $\lambda = (\sum_{i=1}^n \alpha_i, \sum_{i=1}^{n-1} \alpha_i, \dots, \alpha_1)$ *and* $\mu + \nu = \alpha'$. *Every T in* $\mathrm{SSYT}_\lambda\{1,\dots,n\}$ *has a with content* ν.

Proof. It suffices to prove the statement with $\nu = \alpha'$ since if we find a generalized horizontal strip (GHS) with content α' we may forget some entries to obtain a GHS with content ν. We will give an algorithm that is perhaps reminiscent of *row insertion* from [28, Ch.1]. Consider the columns of T in blocks of equal heights $n, n-1, n-2, \dots, 1$ and respective widths $\alpha_1, \dots, \alpha_n$.

We will induct on the blocks of T, showing how to find a GHS with content $(\alpha_1,\dots,\alpha_j)$ in the first j blocks of T. Make a preliminary choice of α_1 $\boxed{1}$'s from the first block, each column of which is full since its height is n and there are only n choices of letters. Moreover,

by the same reasoning we may select any GHS with content $(\beta_1, \beta_2, \dots, \beta_n)$ from the first block whenever $\sum_i \beta_i \le \alpha_1$ and $\beta_i \in \mathbb{N}$, i.e. whenever β is dominated by (α_1).

Next, let β_2 denote the number of $\boxed{2}$'s in the second block (which consists of α_2 columns of height $n-1$). If $\beta_2 = \alpha_2$, choose α_2 $\boxed{2}$'s from the second block. If $\beta_2 < \alpha_2$, some columns in this block are missing a $\boxed{2}$'s, but they can't also be missing $\boxed{1}$'s since the height is $n-1$. Since $\alpha_1 \ge \alpha_2$ we can swap the choices of $\boxed{1}$'s in the first block with $\boxed{2}$'s as many times as necessary until the first block has a chosen GHS with content $(\alpha_1 - (\alpha_2 - \beta_2), (\alpha_2 - \beta_2))$ and the second block has content $(\alpha_2 - \beta_2, \beta_2)$. Now the first two blocks have a GHS with content (α_1, α_2) up to reordering.

For the induction step, suppose that whenever γ is dominated by $(\alpha_1, \dots, \alpha_{j-1})$ we can find a GHS with content γ in the first $j-1$ columns of the tableau. We claim that we can choose any GHS of content $(\beta_1, \dots, \beta j)$ whenever $(\beta_1, \dots, \beta_j)$ is dominated by $(\alpha_1, \dots, \alpha_j)$.

First we attempt to find a GHS with content j^{α_j} in the jth block, since if this is possible the induction hypothesis implies that we can find a GHS with content $(\alpha_1, \dots, \alpha_{j-1})$ in the first $j-1$ blocks. In the j-th block, let β_j denote the number of $\boxed{j}$'s occurring. If $\beta_j = \alpha_j$ we are done. If any column in the jth block is missing a $\boxed{j}$, trade the choice of a $\boxed{j}$ with the next smaller entry in the same column. Since in the jth block each column can be missing at most $j-1$ letters, it can't be missing all letters in $\{1, \dots, j\}$, so we may exchange up to $\alpha_j \le \alpha_{j-1}$ choices of letters smaller than $\boxed{j}$ in the jth block with the previous block. Let γ_i denote the number of $\boxed{i}$'s in the chosen in block j. Then there is a GHS with content $(\gamma_1, \dots, \gamma_{j-1}, \beta_j)$ in block j with $\beta_j + \sum_{i=1}^{j-1} \gamma_i = \alpha_j$. Now we must find a GHS with content $(\alpha_1 - \gamma_1, \dots, \alpha_{j-1} - \gamma_{j-1}, \alpha_j - \beta_j)$ in the first $j-1$ blocks. This is possible by induction because this content is dominated by $(\alpha_1, \dots, \alpha_{j-1})$ since

$$\begin{array}{rcl}
\alpha_1 - \gamma_1 & \le & \alpha_1 \\
\alpha_1 - \gamma_1 + \alpha_2 - \gamma_2 & \le & \alpha_1 + \alpha_2 \\
 & \vdots & \\
\sum_{i=1}^{j-1} (\alpha_i - \gamma_i) & \le & \sum_{i=1}^{j-1} \alpha_i \\
\sum_{i=1}^{j-1} (\alpha_i - \gamma_i) + \alpha_j - \beta_j & = & \sum_{i=1}^{j-1} \alpha_i.
\end{array}$$

Therefore, we've found a GHS with content $(\alpha_1 - \gamma_1, \dots, \alpha_{j-1} - \gamma_{j-1}, \alpha_j - \beta_j)$ in the first $j-1$ blocks and content $(\gamma_1, \dots, \gamma_{j-1}, \beta_j)$ in block j, and (up to re-ordering) we've found a GHS with content $(\alpha_1, \dots, a_j)$ in the first j blocks. □

Example 2.5. Let $\lambda = (12, 11, 8, 5)$, $\alpha = (5, 5, 3, 3, 1)$. To avoid restarts, it makes sense to traverse the algorithm from the last column to the first. We will also keep track of a recording word, which gives the recipe for finding the GHS. Here is an element of $SSYT_\lambda\{1, \dots, 4\}$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1&1&1&1&1&1&1&2&2&2&2&3\\ \hline
2&2&2&2&2&2&3&3&4&4&4\\ \cline{1-11}
3&3&3&3&3&3&4&4\\ \cline{1-8}
4&4&4&4&4\\ \cline{1-5}
\end{array} \qquad \emptyset$$

Let's find a GHS with content $\alpha' = (5, 3, 3, 1)$.

We choose the smallest content possible in the last block, which is a $\boxed{3}$. We choose the smallest content possible in the second to last block, which is $\boxed{2\,|\,2\,|\,2}$.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1&1&1&1&1&1&1&2&2&2&2&3\\ \hline
2&2&2&2&2&2&3&3&4&4&4\\ \cline{1-11}
3&3&3&3&3&3&4&4\\ \cline{1-8}
4&4&4&4&4\\ \cline{1-5}
\end{array} \qquad 2223$$

It remains to find a GHS with content $(5,0,2,1)$ in the first two blocks. The smallest content from the remaining in the 2nd block is $\boxed{1\,|\,1\,|\,3}$:

$$\begin{array}{cccccccccccc}
1&1&1&1&1&\colorbox{cyan}{1}&\colorbox{cyan}{1}&2&\colorbox{cyan}{2}&\colorbox{cyan}{2}&\colorbox{cyan}{2}&\colorbox{cyan}{3}\\
2&2&2&2&2&2&3&\colorbox{cyan}{3}&4&4&4&\\
3&3&3&3&3&3&4&4&&&&\\
4&4&4&4&4&&&&&&&
\end{array}\qquad 1132223$$

Finally, we choose the remaining content $(3,0,1,1)$ from the first block:

$$\begin{array}{cccccccccccc}
1&1&\colorbox{cyan}{1}&\colorbox{cyan}{1}&\colorbox{cyan}{1}&\colorbox{cyan}{1}&\colorbox{cyan}{1}&2&\colorbox{cyan}{2}&\colorbox{cyan}{2}&\colorbox{cyan}{2}&\colorbox{cyan}{3}\\
2&2&2&2&2&2&3&\colorbox{cyan}{3}&4&4&4&\\
3&\colorbox{cyan}{3}&3&3&3&3&4&4&&&&\\
\colorbox{cyan}{4}&4&4&4&4&&&&&&&
\end{array}\qquad 431111132223$$

We've found a GHS with the desired content.

For curiosity we can replace all selected content with 0's and shuffle the 0's upward to produce

$$\begin{array}{cccccccccccc}
1&1&0&0&0&0&0&2&0&0&0&0\\
2&2&2&2&2&2&3&0&4&4&4&\\
3&0&3&3&3&3&4&4&&&&\\
0&4&4&4&4&&&&&&&
\end{array}\quad\text{and}\quad
\begin{array}{cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&2&2&2&2&3&2&4&4&4&\\
2&2&3&3&3&3&4&4&&&&\\
3&4&4&4&4&&&&&&&
\end{array},$$

the last of which happens to be almost semi-standard. $\diamondsuit$

2.7. **Generalized horizontal strip placing.** We will be interested in the opposite of GHS-finding in a tableau: find which tableaux arose from adding a GHS with a given content. The opposite operation is related to the opposite tableau and the opposite filling.

Recall that if λ is a partition, $b \geq \lambda_1$ and $a \geq \#\lambda$, the *contragradient partition with respect to an $a \times b$ box*, denoted λ^* when a and b are understood, is the partition with parts $b - \lambda_i$ for $a \geq i \geq 1$. The name comes from the fact that the representation $S_\lambda(V^*)$ is isomorphic to $S_{\lambda^*}V$ up to a twist by an integer power of $\bigwedge^{\dim V} V$. An exact duality can be seen in bases as follows. Suppose $T \in \mathrm{SSYT}_\lambda\{0,\ldots,n\}$, and let T_i denote column i of T. Define T^* as the tableau whose columns are $T^*_{a-i} = \{0,\ldots,n\} - T_i$ for (this is the Hodge dual with respect to the standard volume form). The resulting tableau T^* is necessarily semi-standard.

Proposition 2.6. *Suppose π is a partition with n parts and let $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| = d$. If π^* dominates a partition γ that is monomial-optimal for α then a GHS with content α can be added to every $T \in \mathrm{SSYT}_\pi\{0,\ldots,n\}$ producing (after shuffling) a non-zero linear combination of semi-standard tableaux of shape (d,π).*

Proof. First note that if $Y \in \mathrm{SSYT}_{(n+1)^d}\{0,\ldots,n\}$, then Y contains every GHS with content $\alpha \in \{0,\ldots,n\}^d$. So we can add a GHS to $T \in \mathrm{SSYT}_\pi\{0,\ldots,n\}$ with content α and no repeats in any columns if and only if we can find a GHS with content α in T^*. For this we apply Lemma 2.4 to T^*. The fact that a tableau with no repeats in any columns is nonzero follows from Lemma 2.7. $\square$

Lemma 2.7. *Suppose T is a filled tableau of shape λ and has no repeats in any columns. Then applying straightening rules to T will produce a* non-zero *linear combination of semi-standard tableaux.*

Proof. From T construct the contragradient filling T^* by considering the missing content in each column. Concatenate T and T^* (rotated 180 degrees) into an $a \times (n+1)$ box B. Since no column of B has any repeats, column shuffles of B will produce $\pm B$ with every column with full content $\{0,\ldots,n\}$ in strictly increasing order, which is in turn a non-zero semi-standard tableaux. This shows that the standard pairing of T and T^* produce a non-zero element of

$\bigwedge^{n+1} V^a$, so both T and T^* must be non-zero in their respective vector spaces by the linearity and non-degeneracy of the standard pairing. $\square$

3. Young flattenings

In the study of tensors, a flattening is a construction that produces a matrix from a tensor. A symmetric flattening of a homogeneous polynomial $\varphi \in S^d V$ is obtained from the natural inclusion $S^d V \hookrightarrow S^k V \otimes S^{d-k} V$ (the standard comultiplication map). Let C^k_φ denote the image of this inclusion. After a choice of basis C^k_φ may be represented by a matrix (either as a linear map $S^k V^* \to S^{d-k} V$ or its transpose). Symmetric flattenings are also often called catalecticants, generalized Hankel matrices, or moment matrices depending on the context. Landsberg and Ottaviani's Young flattenings [42] are higher order flattenings for tensors.

The tensor product of two irreducible representations $S_\lambda V$ and $S_\pi V$ decomposes as a sum of irreducible representations, whose combinatorics are governed by the Littlewood-Richardson rule. In the case that $\lambda = (d)$ we have the so called Pieri rule or Pieri product:

$$S_{(d)} V \otimes S_\pi V = \bigoplus_{\mu - \pi \in HS(d)} S_\mu V, \tag{2}$$

where the sum is over all partitions μ obtained from π by adding d boxes, no two in the same column. Equivalently the sum is over the partitions μ such that the difference $\mu - \pi$ is a GHS of width d. And yet another way to think of this sum is that it is over those partitions μ that are obtained from π by adding a GHS. Each irreducible representation in the direct sum in (2) occurs at most once, i.e. the decomposition is *multiplicity free*.

The projection $S_{(d)} V \otimes S_\pi V \to S_\mu V$ (defined by (2)) may be interpreted as a bilinear map

$$S_{(d)} V \times S_\pi V \to S_\mu V,$$

or as a linear map depending linearly on $S^d V$

$$S_\pi V \to S_\mu V.$$

Definition 3.1. Suppose λ and μ are partitions such that $S_\lambda V \otimes S^d V \supset S_\mu V$. Let $\mathcal{F}_{\lambda,\mu} \in S^d V^* \otimes S_\lambda V^* \otimes S_\mu V$ denote the restriction of the (labeled) Pieri product $S_\lambda V \otimes S^d V$ to $S_\mu V$. Given $\varphi \in S^d V$, the linear map $\mathcal{F}_{\lambda,\mu}(\varphi) \colon S_\lambda V \to S_\mu V$ is called the Young flattening of φ.

We may obtain a matrix representing $\mathcal{F}_{\lambda,\mu}(\varphi)$ using distinguished bases; the semi-standard tableaux bases for $S_\pi V$ and $S_\mu V$, and the monomial basis $\{x^\alpha = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid |\alpha| = d\}$ for $S^d V$. The labeled Pieri rule is an expression of the usual Pieri rule in these bases.

Definition 3.2. Suppose T_π is a filled tableau of shape $\pi \vdash d$, and let α denote the tableau of shape (d), both with content α in $\{0, \ldots, n\}$. The labeled Pieri rule for the tensor product $T_\pi \otimes \alpha$ is

$$T_\pi \otimes \alpha = \sum Y_\mu,$$

where the sum is over all tableaux Y_μ such that the filling of shape μ is obtained by adding the labeled boxes from α in a fixed order to the outside of T_π with no two boxes in the same column, and every reordering of the entries of α shows up (possibly redundantly) in the sum. After applying shuffling rules (Proposition 2.3), collect terms and write

$$T_\pi \otimes \alpha = \sum_{Y_\mu \in \mathrm{SSYT}(\mu)} m_{T_\pi, Y_\mu}(\alpha) Y_\mu, \tag{3}$$

for some integers $m_{T_\pi,Y_\mu}(\alpha)$. For fixed μ, the matrix $(m_{T_\pi,Y_\mu}(\alpha))$ represents the Young flattening $\mathcal{F}_{\pi,\mu}(x^\alpha)$. If $\varphi = \sum_{\alpha\vdash d}\varphi_\alpha x^\alpha$ is arbitrary in S^dV, the linearity of the construction gives the Young flattening $\mathcal{F}_{\pi,\mu}(\varphi)$ and a matrix representing it.

Let $\lrcorner\alpha$ denote the contraction map, that sends a tableau to the sum of all tableaux gotten by removing a GHS with content α. Note, Olver [48] and Sam [55] use this transposed notion for their description of the Pieri rule. We will most often use the contraction $\lrcorner 0^d$.

Example 3.3. The labeled Pieri product of $\begin{ytableau}0&0\\1\end{ytableau}$ and $\begin{ytableau}2&2&2\end{ytableau}$ is

$$\begin{ytableau}0&0\\1\end{ytableau} \otimes \begin{ytableau}2&2&2\end{ytableau} = 6\cdot\left(\begin{ytableau}0&0&2&2&2\\1\end{ytableau} + \begin{ytableau}0&0&2&2\\1&2\end{ytableau} + \begin{ytableau}0&0&2&2\\1\\2\end{ytableau} + \begin{ytableau}0&0&2\\1&2\\2\end{ytableau}\right).$$

The labeled Pieri product of $\begin{ytableau}0&0\\1\end{ytableau}$ and $\begin{ytableau}0&1&2\end{ytableau}$ is

$$\begin{ytableau}0&0\\1\end{ytableau} \otimes \begin{ytableau}0&1&2\end{ytableau} = 6\cdot\begin{ytableau}0&0&0&1&2\\1\end{ytableau} + 2\cdot\left(\begin{array}{c}\begin{ytableau}0&0&0&1\\1&2\end{ytableau}\\ + \\ \begin{ytableau}0&0&0&2\\1&1\end{ytableau}\end{array}\right) + 2\cdot\begin{ytableau}0&0&0&1\\1\\2\end{ytableau} + \begin{ytableau}0&0&0\\1&1\\2\end{ytableau},$$

where we have deleted those tableaux with repeats in the columns and we have collected redundancies due to permutations of columns of the same size. $\diamondsuit$

3.1. **Young flattenings and bounds on border rank.** The following is a consequence of the additivity of the construction, and sub-additivity and semi-continuity of matrix rank.

Proposition 3.4. [42, Prop.4.1] *Consider $[x^d] \in \nu_d\mathbb{P}V$ and construct the Young flattening $\mathcal{F}_{\lambda,\mu}(x^d)$. Suppose* $\mathsf{Rank}\,\mathcal{F}_{\lambda,\mu}(x^d) = m$*. Then if $[\varphi] \in \mathbb{P}S^dV$ has rank r the Young flattening $\mathcal{F}_{\lambda,\mu}(\varphi)$ has rank at most rm. Thus the $(rm+1)\times(rm+1)$ minors of a generic $\mathcal{F}_{\lambda,\mu}(\varphi)$ provide equations for $\sigma_r(\nu_d\mathbb{P}V)$.*

In our experience, the Young flattenings that are most efficient, i.e. give the best lower bounds on border rank, are those that take tableaux of shape λ and add a row with no gaps producing shape (d,λ). In this case we simplify notation and write $\mathcal{F}_\lambda$ in place of $\mathcal{F}_{(\lambda,(d,\lambda))}$.

It is straightforward to find the multiplier m in Proposition 3.4. Let V_0 denote $V/\langle x_0\rangle$. The following is an easy consequence of Schur's lemma for $\mathfrak{gl}(V_0)$ modules.

Lemma 3.5. *Suppose $d \geq \lambda_0$. The map $\mathcal{F}_\lambda(x_0^d)\colon S_\lambda V \to S_{d,\lambda}V$ satisfies*

$$\mathcal{F}_\lambda(x_0^d)(S_\lambda V) = \mathcal{F}_\lambda(x_0^d)(S_\lambda V_0) \;\cong S_\lambda V_0.$$

In particular, $\mathcal{F}_\lambda(x_0^d)$ has rank $\dim S_\lambda V_0$.

Proof. Any tableau that has content with a $\boxed{0}$ will get sent to 0 by adding a row of zeros. So we restrict the map to $S_\lambda V_0$. Consider the subspace $\mathcal{B}_{d,\lambda}(0^d)$ of $S_{(d,\lambda)}V$ spanned by tableaux of shape (d,λ) with all $\boxed{0}$'s in the first row. The bijection of this spanning set with the basis $\mathrm{SSYT}_\lambda\{1,\ldots,n\}$ of $S_\lambda V_0$ is by removing d $\boxed{0}$'s from the unique such horizontal strip in each tableau in a basis of $\mathcal{B}_{d,\lambda}(0^d)$.

Another way to check this result is to see that $\mathcal{F}_\lambda(x_0^d)$ is a $\mathfrak{gl}(V_0)$-module homomorphism, and it restricts to an isomorphism

$$S_\lambda V_0 \to \mathcal{B}_{d,\lambda}(0^d)$$

(by checking that it is non-zero on this module, and applying Schur's lemma). $\square$

Remark 3.6. Because matrix rank is lower semi-continuous, if we have $\mathsf{Brank}\ (\varphi) \leq r$, then necessarily $\mathsf{Rank}\ \mathcal{F}_{\lambda,\mu}(\varphi) \leq rm$. That is, an upper bound on the border rank of a form also induces an upper bound on the rank of any Young flattening of that form. On the other hand, non-vanishing minors of Young flattenings are certificates of lower bounds for border rank. Showing that these equations are actually non-trivial can be challenging.

It will turn out that a special type of partition (monomial-optimal) will be most useful.

Definition 3.7 (Monomial-optimal shapes)**.** Suppose $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$, $d = \sum_{i=0}^n \alpha_i$. Consider

$$\lambda = \left(\sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n-1} \alpha_i, \sum_{i=1}^{n-2} \alpha_i, \ldots, \alpha_2 + \alpha_1, \alpha_1, 0 \right).$$

We say that λ is α-$\mathsf{optimal}$, and it will turn out that the flattening $\mathcal{F}_\lambda$ will be optimal for finding lower bounds for the border rank of x^α.

Remark 3.8. The optimal shape has another interpretation via duality of $GL(V)$-representations. Let ω_i denote the fundamental weights for representations of $GL(V)$, and let Γ_α denote the irreducible $GL(V)$-representation with highest weight $\alpha_0\omega_0 + \cdots + \omega_n\alpha_n$. The optimal shape λ has weight $\alpha_n\omega_1 + \alpha_{n-1}\omega_2 + \cdots + \alpha_1\omega_n$, which is the weight of the dual to the weight module Γ_α. That is $\Gamma_\alpha = (S_\lambda V)^*$, which is also known as the *contragradient* representation.

4. Properties of (partial) Young flattenings

4.1. Extra symmetry.

If the form φ has symmetry, $\mathcal{F}_{\lambda,\mu}(\varphi)$ inherits this symmetry.

Proposition 4.1. *Consider the Young flattening $\mathcal{F}_{\lambda,\mu}(\varphi) \colon S_\lambda V \to S_\mu V$. Suppose G is a subgroup of $\mathrm{GL}(V)$ that fixes φ, then the Young flattening $\mathcal{F}(\varphi)$ is G-equivariant. In particular, the Young flattening $\mathcal{F}_{\lambda,\mu}(x^\alpha)$ is equivariant for the action of $\mathbb{T}^{n+1} \rtimes \mathrm{Fix}_\alpha(\mathfrak{S}_{n+1})$.*

Proof. It was already proved in [42, Proposition 4.1] that $\mathcal{F}_{\lambda,\mu}(\varphi)$ is a linear map of vector spaces. Note that $S_\lambda V$ and $S_\mu V$ are $\mathrm{GL}(V)$-modules so they are G-modules for any $G < \mathrm{GL}(V)$. Recall that $\mathcal{F}_{\lambda,\mu}$ is the restriction of the labeled Pieri product (3)

$$\mathcal{F}_{\lambda,\mu}(\varphi)(T_\pi) = T_\pi \otimes \varphi,$$

where we regard φ as the semi-standard tableau with content equal to the indices which occur in the monomial φ, namely, if $\varphi = x_{i_0} \cdots x_{i_n}$, and $i_0 \leq \cdots \leq i_n$ then we identify φ with $\boxed{i_0}\boxed{i_1}\ \ldots\ \boxed{i_n}$. If φ is a linear combination of monomials we extend the definition by linearity. For $g \in \mathrm{GL}(V)$ we have

$$g.(\mathcal{F}_{\lambda,\mu}(\varphi)(T_\pi)) = g.(T_\pi \otimes \varphi) = (g.T_\pi) \otimes (g.\varphi). \tag{4}$$

Now if $g.\varphi = \varphi$ then

$$g.(T_\pi \otimes \varphi) = (g.T_\pi) \otimes (g.\varphi) = (g.T_\pi) \otimes \varphi.$$

Thus

$$g.(\mathcal{F}_{\lambda,\mu}(\varphi)(T_\pi)) = \mathcal{F}_{\lambda,\mu}(\varphi)(g.T_\pi)$$

for all $g \in G$ and for all $T_\pi \in \mathrm{SSYT}(\pi)$. So the Young flattening $\mathcal{F}_{\lambda,\mu}(\varphi)$ is G-equivariant. $\square$

4.2. **The Lie algebra action on Young flattenings.** We want to understand the Lie algebra action of $\mathfrak{gl}(V)$ on Young flattenings. Suppose $\varphi \in S^d V$, and regard φ as a linear combination of monomials, where we reinterpret each monomial $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ as a single row semi-standard Young tableau with content $(0^{\alpha_0}, \dots, n^{\alpha_n})$.

Proposition 4.2. *Fix $\varphi \in S^d V$ and let $\mathcal{F} = \mathcal{F}_{\lambda,\mu}$ be a Young flattening. For $X \in \mathfrak{gl}(V)$ and $T \in S_\lambda V$ we have*

$$\mathcal{F}(X.\varphi)(T) = \mathcal{F}(\varphi)(X.T) - X.(\mathcal{F}(\varphi)(T)). \tag{5}$$

Proof. The result follows from basic algebra and an application of $X \in \mathfrak{gl}(V)$:

$$\begin{aligned} X.(\mathcal{F}(\varphi)(T)) &= X.(T \otimes \varphi) \\ &= (X.T) \otimes \varphi + T \otimes (X.\varphi) \\ &= \mathcal{F}(\varphi)(X.T) + \mathcal{F}(X.\varphi)(T). \quad \square \end{aligned}$$

We will need a partial action of $\mathfrak{gl}(V)$ on Young flattenings. For each $X \in \mathfrak{gl}(V)$, $T \in S_\lambda V$ and $\varphi \in S^d V$ define the **partial Young flattening** as

$$\begin{array}{rcccc} \mathcal{F}^p(X.\varphi)\colon S_\lambda V & \to & S_{d,\lambda} V & \oplus & S_{d,\lambda} V \otimes \mathfrak{gl}(V) \\ T & \mapsto & \mathcal{F}(\varphi)(X.T) & - & (\mathcal{F}(\varphi)(T)) \otimes X. \end{array} \tag{6}$$

Remark 4.3. From the definition of the partial flattening, (6), the linearity of $\mathcal{F}$, the linearity of the $\mathfrak{gl}(V)$-action and the linearity of the tensor product, it is clear that $\mathcal{F}^p(X.\varphi)$ is a linear mapping depending linearly on φ.

We would like to iterate the application of elements of $\mathfrak{gl}(V)$, and for this it is simpler to work with the universal enveloping algebra of $\mathfrak{gl}(V)$, denoted $\mathcal{U}(\mathfrak{gl}(V))$. Let $X^\gamma = (X_1)^{\gamma_1} \cdots (X_t)^{\gamma_t}$ be an element of $\mathcal{U}(\mathfrak{gl}(V))$ with X_i linearly independent elements of $\mathfrak{gl}(V)$. By iterating (6) we obtain a mapping (partial Young flattening)

$$\begin{array}{rcl} \mathcal{F}^p(X^\gamma.\varphi)\colon S_\lambda V & \to & \displaystyle\bigoplus_{\mu+\nu=\gamma} S_{d,\lambda} V \otimes \langle X^\nu \rangle \subset S_{d,\lambda} V \otimes \mathfrak{gl}(V) \\ T & \mapsto & \displaystyle\sum_{\mu+\nu=\alpha'} (-1)^{|\nu|} (\mathcal{F}(\varphi)(X^\mu.T)) \otimes X^\nu. \end{array} \tag{7}$$

The following is a key to obtaining the lower bound on the border rank of all monomials.

Proposition 4.4. *The partial Young flattening $\mathcal{F}^p(X^\gamma.\varphi)$ has rank*

$$\sum_{\mu+\nu=\gamma} \dim \operatorname{span}\{\mathcal{F}(\varphi)(X^\mu.T) \mid T \in S_\lambda V\}.$$

Proof. By construction $\bigoplus_{\mu+\nu=\gamma} \operatorname{span}\{\mathcal{F}(\varphi)(X^\mu.T) \mid T \in S_\lambda V\} \otimes \langle X^\nu \rangle$ is the image of $\mathcal{F}^p(X^\gamma.\varphi)$. The rank follows from the fact that these vector spaces form a direct sum and the fact that the set $\{X^\nu \mid \nu \preceq \gamma\}$ is linearly independent in $\mathcal{U}(\mathfrak{gl}(V))$. $\square$

In the usual Young flattening case, iterating Proposition 4.2 we obtain

$$\begin{array}{rcl} \mathcal{F}(X^\gamma.\varphi)\colon S_\lambda V & \to & \displaystyle\sum_{\mu+\nu=\gamma} X^\nu.S_{d,\lambda} V \subset S_{d,\lambda} V \\ T & \mapsto & \displaystyle\sum_{\mu+\nu=\alpha'} (-1)^{|\nu|} X^\nu(\mathcal{F}(\varphi)(X^\mu.T)). \end{array} \tag{8}$$

In some cases, the target of the usual Young flattening is not a direct sum of vector subspaces of $S_{d,\lambda}V$ each isomorphic to $S_\lambda V_0$, and thus the flattening will have smaller rank than the partial flattening of the same type. In the following section we will describe precisely when this behavior occurs. But first let us explain in more detail how special elements of $\mathcal{U}(\mathfrak{gl}(V))$ interact with Young flattenings of monomials.

Let X_i^j denote the element of the $\mathfrak{gl}(V)$ that acts as $\boxed{j} \cdot \frac{\partial}{\partial \boxed{i}}$ on tableaux, that is, X_i^j removes a box $\boxed{i}$ and replaces it with a box $\boxed{j}$ and acts as a derivation. Let $X_0^\gamma := (X_0^n)^{\gamma_n}(X_0^{n-1})^{\gamma_{n-1}} \cdots (X_0^1)^{\gamma_1}$. Let H_0 denote the linear map constructed by composing row addition and row subtraction:

$$H_0(T) := (\mathcal{F}_\lambda(0^d)(T)) \lrcorner 0^d.$$

Lemma 3.5 essentially says that H_0 acts as the identity on $S_\lambda V_0$ and the quotient $S_\lambda V / S_\lambda V_0$ is isomorphic to the kernel of H_0.

Example 4.5. Consider the monomial $x_0^{d-1}x_1$, the flattening $\mathcal{F}(x_0^{d-1}x_1)\colon S^1V \to S_{d,1}V$, and the Lie algebra element X_0^1 that switches a 0 for a 1. Let V_0 denote the quotient $V/\langle x_0\rangle$. Notice that $V_0 = \ker(X_0^1)$ and $\langle x_0\rangle = \ker(\mathcal{F}(x_0^d))$ are complementary subspaces of V. So if we chose T in one or the other of $\langle x_0\rangle$ or V_0 at most one term on the right hand side of

$$\mathcal{F}(X_0^1.x_0^d)(T) = \mathcal{F}(x_0^d)(X_0^1.T) - X_0^1.(\mathcal{F}(x_0^d)(T))$$

is non-zero.

If $T' \in V_0$, and $T'' \in \langle x_0\rangle$ and $T = T' + T''$ then

$$\begin{aligned} \mathcal{F}(X_0^1.x_0^d)(T) &= \mathcal{F}(x_0^d)(X_0^1.T) - X_0^1.(\mathcal{F}(x_0^d)(T)) \\ &= \mathcal{F}(x_0^d)(X_0^1.T'') - X_0^1.(\mathcal{F}(x_0^d)(T')). \end{aligned}$$

Now notice that $\mathcal{F}(x_0^d)(X_0^1.T'')$ and $-X_0^1.(\mathcal{F}(x_0^d)(T'))$ have different content, so they live in disjoint vector subspaces of $S_{d,1}V$. This means that the mapping $\mathcal{F}(x_0^{d-1}x_1) = \mathcal{F}(X_0^1.x_0^d)$ splits as a direct sum:

$$\begin{tikzcd}
 & & \langle x_1\rangle \arrow[dr, "x_0^d"] & & \\
 & \langle x_0\rangle \arrow[ur, "X_0^1"] & & \big(\langle x_1\rangle \otimes \langle x_0^d\rangle\big)_{d,1} & \\
V = & \oplus \arrow[rr, "x_0^{d-1}x_1"] & & \oplus & \subset S_{d,1}V \\
 & V_0 \arrow[dr, "x_0^d"'] & & \big(V_0 \otimes \langle x_0^{d-1}x_1\rangle\big)_{d,1} & \\
 & & \big(V_0 \otimes \langle x_0^d\rangle\big)_{d,1} \arrow[ur, "X_0^1"'] & &
\end{tikzcd}$$

Moreover, in this case all the maps in question are isomorphisms, so the image has dimension $1 + \dim V_0 = n + 1$. $\diamondsuit$

Example 4.6. Consider the monomial $x_0^2x_1x_2$, the flattening $\mathcal{F}(x_0^4x_1x_2)\colon S_{2,1}V \to S_{4,2,1}V$, and the Lie algebra elements X_0^1 and X_0^2. Let $V = \mathbb{C}^3$, and let V_0 denote the quotient $V/\langle x_0\rangle$.

We want to use the expression

$$\begin{aligned} \mathcal{F}(X_0^2X_0^1.(x_0)^4)(T) = \quad & \mathcal{F}((x_0)^4)(X_0^2X_0^1.T) - X_0^1.(\mathcal{F}((x_0)^4)(X_0^2T)) \\ & -X_0^2.(\mathcal{F}((x_0)^4)(X_0^1T)) + \mathcal{F}((x_0)^4)(X_0^2X_0^1.T) \end{aligned} \tag{9}$$

For each $\gamma \subset \{1, 2\}$, we obtain a direct sum decomposition

$$S_{2,1}V = \mathcal{A}(\gamma) \oplus \mathcal{K}(\gamma),$$

with $\mathcal{K}(\gamma) = \ker H_0 X_0^\gamma \colon S_{2,1}V \to S_{2,1}V$, and $\mathcal{A}(\gamma) \cong S_{2,1}V/\mathcal{K}(\gamma) \subset S_{2,1}V$. One can check (**claim 1**) $\mathcal{A}(\gamma)$ is isomorphic to $S_\lambda V_0$ for each $\gamma \subset \{1,2\}$, and moreover that (**claim 2**)

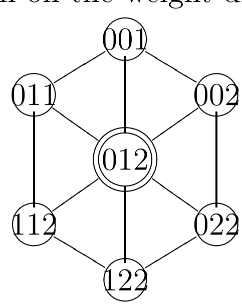
$$S_{2,1}V = \mathcal{A}(1,2) \oplus \mathcal{A}(2) \oplus \mathcal{A}(1) \oplus \mathcal{A}(\emptyset).$$

Claims 1 and 2 can be proven by verifying the following identities:

$$\mathcal{A}(1,2) = \left\langle \ytableaushort{00,1}, \ytableaushort{00,2} \right\rangle,\ \mathcal{A}(1) = \left\langle \ytableaushort{01,2}, \ytableaushort{02,2} \right\rangle,\ \mathcal{A}(2) = \left\langle \ytableaushort{01,1}, \ytableaushort{02,1} \right\rangle,\ \mathcal{A}(\emptyset) = \left\langle \ytableaushort{11,2}, \ytableaushort{12,2} \right\rangle.$$

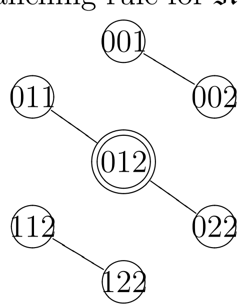
For example, consider $H_0X_0^1X_0^2 \colon S_{2,1}V \to S_{2,1}V$. Any tableau with less than two 0's is in the kernel of $X_0^1X_0^2$. After applying $X_0^1X_0^2$ to $S_{2,1}V$, we get $\left\langle \ytableaushort{11,2}, \ytableaushort{12,2} \right\rangle$, which does not intersect the kernel of H_0. So $\mathcal{K}(1,2) = \left\langle \ytableaushort{00,1}, \ytableaushort{00,2} \right\rangle^{\perp *}$. Now consider $H_0X_0^1 \colon S_{2,1}V \to S_{2,1}V$. Note that $\mathcal{A}(1,2)$ is in the kernel of $H_0X_0^1$ because after applying X_0^1 all vectors in this space have content with one 0, and as such are annihilated by H_0. $\mathcal{A}(2)$ is in the kernel of X_0^1 since the basis vectors are sent to zero (every column with a 0 also has a 1). $\mathcal{A}(\emptyset)$ is in the kernel of X_0^1 since neither basis vector has a 0 in its content. It remains to see that $\mathcal{A}(1)$ does not intersect the kernel of $H_0X_0^1$, which follows from the fact that $H_0X_0^1\mathcal{A}(1) = S_{2,1}V_0$. The other cases are similar.

We may visualize this decomposition on the weight diagram for $S_{2,1}\mathbb{C}^3$:



The double circle indicates that the weight space with weight (012) is two-dimensional.

A standard branching rule for $\mathfrak{sl}(3) \downarrow \mathfrak{sl}(2)$ would decompose $S_{2,1}\mathbb{C}^3$ into $\mathfrak{sl}(2)$ modules as:



$$, \quad \text{or} \quad S_{2,1}\mathbb{C}^3 = S_{2,1}\mathbb{C}^2 \oplus S_2\mathbb{C}^2 \oplus S_1\mathbb{C}^2,$$

where the connected components in the diagram indicate different $\mathfrak{sl}(2)$ modules.

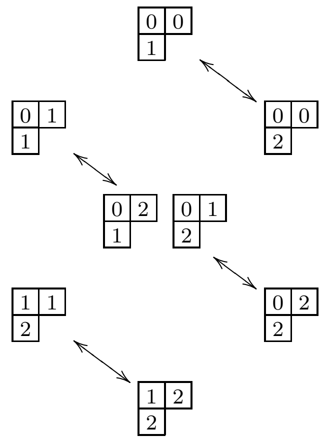
However, we need a different type of branching:

$$S_{2,1}\mathbb{C}^3 = \mathcal{A}(1,2) \oplus \mathcal{A}(2) \oplus \mathcal{A}(1) \oplus \mathcal{A}(\emptyset) = S_{2,1}\mathbb{C}^2 \oplus S_{2,1}\mathbb{C}^2 \oplus S_{2,1}\mathbb{C}^2 \oplus S_{2,1}\mathbb{C}^2,$$

where

$$\mathcal{A}(1,2) = \left\langle \ytableaushort{00,1}, \ytableaushort{00,2} \right\rangle,\ \mathcal{A}(1) = \left\langle \ytableaushort{01,2}, \ytableaushort{02,2} \right\rangle,\ \mathcal{A}(2) = \left\langle \ytableaushort{01,1}, \ytableaushort{02,1} \right\rangle,\ \mathcal{A}(\emptyset) = \left\langle \ytableaushort{11,2}, \ytableaushort{12,2} \right\rangle.$$

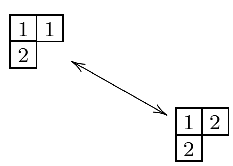
We visualize this decomposition as



The arrows $\longleftrightarrow$ indicate the $\mathfrak{sl}(2)$-modules with the (nonstandard) action of $\mathfrak{sl}(2) = \langle X_1^2, X_2^1, H_{12}\rangle$ given by passing the algebra action through the base $\left\langle \begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \right\rangle$. We can visualize the construction of each $\mathfrak{sl}(2)$ module by applying elements of $\mathcal{U}(\mathfrak{gl}(V))$ to move the weight diagram, those weight spaces that end up outside the weight diagram for $S_{2,1}\mathbb{C}^3$ are in the kernel of the action.

For example, $H_0X_0^1X_0^2$ moves the weight diagram down two steps, and the only remaining weights are $\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}$. So the only weights not in the kernel of $H_0X_0^1X_0^2$ is the top part of the diagram, $\left\langle \begin{ytableau} 0 & 0 \\ 1 \end{ytableau}, \begin{ytableau} 0 & 0 \\ 2 \end{ytableau} \right\rangle = \mathcal{A}(1,2)$. In the $\mathcal{A}(2)$ case, X_0^1 moves the diagram south-west one step, the result is supported on the south-west part of the weight lattice:

$$-\begin{ytableau} 0 & 2 \\ 1 \end{ytableau} + 2\begin{ytableau} 0 & 1 \\ 2 \end{ytableau}$$



Now the action of H_0 annihilates $-\begin{ytableau} 0 & 2 \\ 1 \end{ytableau} + 2\begin{ytableau} 0 & 1 \\ 2 \end{ytableau}$. We recall that $\left\langle \begin{ytableau} 0 & 1 \\ 1 \end{ytableau}, \begin{ytableau} 0 & 2 \\ 1 \end{ytableau} \right\rangle$ mapped to $\left\langle \begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \right\rangle$, and is a complement of the kernel of $H_0X_0^1$.

By construction only one term on the right hand side of (9), namely $\mathcal{F}(x_0^4)(X_0^\gamma\cdot)$, is non-zero on each vector space $\mathcal{A}(\gamma)$, so the expression at (9) splits over the direct sum. By the fact that $H_0X_0^\gamma S_{2,1}V = S_{2,1}V_0$ and Lemma 3.5,

$$\mathcal{F}(x_0^4)(X_0^\gamma\cdot) \colon \mathcal{A}(\gamma) \to S_{4,2,1}V$$

is an isomorphism onto its image, which is $\mathcal{F}_{2,1}(x_0^4)(S_{2,1}V_0) \cong S_{2,1}V_0$.

Finally, one checks (**claim 3**) that the vector spaces $\mathcal{B}(\nu) := X_0^\nu\mathcal{F}_{2,1}(x_0^4)(S_{2,1}V_0)$ for $\nu \subset \{1,2\}$ are linearly independent subspaces of $S_{d,2,1}V$. In particular, $S_{4,2,1}\mathbb{C}^3 \supset \mathcal{B}(\emptyset) + \mathcal{B}(1) + \mathcal{B}(2) + \mathcal{B}(1,2)$, with

$$\mathcal{B}(\emptyset) = \left\langle \begin{ytableau} 0 & 0 & 0 & 0 \\ 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 0 & 0 & 0 & 0 \\ 1 & 2 \\ 2 \end{ytableau} \right\rangle, \quad \mathcal{B}(1) = X_0^1 \left\langle \begin{ytableau} 0 & 0 & 0 & 0 \\ 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 0 & 0 & 0 & 0 \\ 1 & 2 \\ 2 \end{ytableau} \right\rangle,$$

$$\mathcal{B}(2) = X_0^2 \left\langle \begin{ytableau} 0&0&0&0\\ 1&1\\ 2 \end{ytableau}, \begin{ytableau} 0&0&0&0\\ 1&2\\ 2 \end{ytableau} \right\rangle, \quad \mathcal{B}(1,2) = X_0^2 X_0^1 \left\langle \begin{ytableau} 0&0&0&0\\ 1&1\\ 2 \end{ytableau}, \begin{ytableau} 0&0&0&0\\ 1&2\\ 2 \end{ytableau} \right\rangle.$$

Moreover, the sum is direct.

So we have a direct sum

$$\mathcal{F}_{2,1}(x_0^4 x_1 x_2)\colon S_{2,1}V = \left\{ \begin{array}{ccc} \mathcal{A}(1,2) & \to & \mathcal{B}(\emptyset) \\ & \oplus & \\ \mathcal{A}(2) & \to & \mathcal{B}(1) \\ & \oplus & \\ \mathcal{A}(1) & \to & \mathcal{B}(2) \\ & \oplus & \\ \mathcal{A}(\emptyset) & \to & \mathcal{B}(1,2) \end{array} \right\} \subset S_{4,2,1}V.$$

Respectively choose ordered bases for the source ($S_{2,1}V$) and target ($S_{4,2,1}V$):

$$\left\{ \begin{ytableau} 0&0\\ 1 \end{ytableau}, \begin{ytableau} 0&0\\ 2 \end{ytableau}, \begin{ytableau} 0&1\\ 1 \end{ytableau}, \begin{ytableau} 0&1\\ 2 \end{ytableau}, \begin{ytableau} 0&2\\ 1 \end{ytableau}, \begin{ytableau} 0&2\\ 2 \end{ytableau}, \begin{ytableau} 1&1\\ 2 \end{ytableau}, \begin{ytableau} 1&2\\ 2 \end{ytableau} \right\},$$

$$\left\{ \begin{array}{c} \begin{ytableau} 0&0&0&0\\ 1&1\\ 2 \end{ytableau}, \begin{ytableau} 0&0&0&0\\ 1&2\\ 2 \end{ytableau}, \begin{ytableau} 0&0&0&1\\ 1&1\\ 2 \end{ytableau}, \begin{ytableau} 0&0&0&1\\ 1&2\\ 2 \end{ytableau}, \begin{ytableau} 0&0&0&2\\ 1&1\\ 2 \end{ytableau}, \begin{ytableau} 0&0&0&2\\ 1&2\\ 2 \end{ytableau}, \begin{ytableau} 0&0&1&1\\ 1&1\\ 2 \end{ytableau}, \\ \begin{ytableau} 0&0&1&1\\ 1&2\\ 2 \end{ytableau}, \begin{ytableau} 0&0&1&2\\ 1&1\\ 2 \end{ytableau}, \begin{ytableau} 0&0&1&2\\ 1&2\\ 2 \end{ytableau}, \begin{ytableau} 0&0&2&2\\ 1&1\\ 2 \end{ytableau}, \begin{ytableau} 0&0&2&2\\ 1&2\\ 2 \end{ytableau}, \begin{ytableau} 0&1&1&1\\ 1&2\\ 2 \end{ytableau}, \begin{ytableau} 0&1&1&2\\ 1&2\\ 2 \end{ytableau}, \begin{ytableau} 0&1&2&2\\ 1&2\\ 2 \end{ytableau} \end{array} \right\}.$$

Our maps are respectively $M_{12,\emptyset}, M_{1,2}, M_{2,1}, M_{\emptyset,12} =$

$$\begin{bmatrix} -1&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&4&-2&0&0&0\\ 0&0&0&0&0&4&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-2&0&0&0&0&0\\ 0&0&0&4&-8&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&12&0\\ 0&0&0&0&0&0&0&12\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

One verifies that $M_{12,\emptyset} - M_{1,2} - M_{2,1} + M_{\emptyset,12} = 12M =$ $$\begin{bmatrix} -1&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0\\ 0&0&0&-4&8&0&0&0\\ 0&0&0&-4&2&0&0&0\\ 0&0&0&0&0&-4&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&12&0\\ 0&0&0&0&0&0&0&12\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

In particular, the image is the vector subspace of $S_{4,2,1}V$ isomorphic to

$$S_{2,1}V_0 \otimes \langle x_0^4, x_0^3 x_1, x_0^3 x_2, x_0^2 x_1 x_2 \rangle.$$

Applying Proposition 3.4, we find that the border rank of $x_0^2 x_1 x_2$ is at least 4; the ratio of the dimensions of $S_{2,1}V_0 \otimes \langle x_0^4, x_0^3 x_1, x_0^3 x_2, x_0^2 x_1 x_2 \rangle$ and $S_{2,1}V_0$, which is also the known upper bound, so the border rank is exactly 4. From this exposition, perhaps it is also clear that x_0^2 may be replaced by x_0^d for any $d \geq 1$ and while the number of rows in the matrices will change, the same bound on the border rank of $x_0^d x_1 x_2$ will be obtained. $\diamondsuit$

In the Section 5 we'll develop the tools necessary to prove the general cases for claims 1-3 in the above example. We will also see when claim 3 is true for Young flattenings and partial Young flattenings. First we describe how to carry out these computations in Macaulay2.

4.3. **Macaulay2 implementation.** We implemented Young flattenings in Macaulay2 [30] using the package PieriMaps developed by Sam [55]. Here is the construction of $\mathcal{F}_{4,3,2,1}$ evaluated at x_0^5 and at $x_0x_1x_2x_3x_4$:

```
loadPackage"PieriMaps"
A = QQ[x_0..x_4]
MX = pieri({5,4,3,2,1},{5,4,3,2,1},A);
diff(x_0^5, MX)
diff(x_0*x_1*x_2*x_3*x_4, MX)
```

The slowest part of the computation is constructing the map MX. We computed the rank of $\mathcal{F}_{4,3,2,1}(\varphi)$ for each monomial φ via the following:

```
L = apply(partitions(5), p-> product(#p, i-> (x_i )^(p_i) ))
for i to #L-1 do  print(toString L_i,rank diff(L_i , MX) );
> (x_0^5, 64)
> (x_0^4*x_1, 128)
> (x_0^3*x_1^2, 176)
> (x_0^3*x_1*x_2, 256)
> (x_0^2*x_1^2*x_2, 324)
> (x_0^2*x_1*x_2*x_3, 512)
> (x_0*x_1*x_2*x_3*x_4, 1024)
```

So this flattening predicts the best lower bound for the border rank of all quintic monomials:

$$\begin{gathered}\mathsf{Brank}\ x_0^4x_1 \geq 2,\\ \mathsf{Brank}\ x_0^3x_1^2 \geq 3,\\ \mathsf{Brank}\ x_0^3x_1x_2 \geq 4,\\ \mathsf{Brank}\ x_0^2x_1^2x_2 \geq 6,\\ \mathsf{Brank}\ x_0^2x_1x_2x_3 \geq 8,\\ \mathsf{Brank}\ x_0x_1x_2x_3x_4 \geq 16.\end{gathered}$$

However, it is curious that the ranks of the flattenings of some monomials are not multiples of their border ranks. We found it difficult to predict for a single type of flattening its rank for each monomial. This curiosity led us to search for optimal shapes of partitions that for a fixed monomial x^α a specific flattening that has the "correct" rank in order to predict the border rank. We list the optimal shapes for monomials of degrees 6 and 7 in Tables 1 and 2.

Larger examples can quickly become computationally intensive. For example, while computing the optimal Young flattenings for $x_0^2x_1^2x_2^2x_3$ took only 234s, but the similar computation for $x_0^3x_1^2x_2^2x_3$ took approximately 8 hours on our server.

Exponent vectors of monomials are (up to reordering variables) partitions, and are partially ordered by dominance. Moreover, the lowering operators, X_i^j with $i < j$, in the Lie algebra $\mathfrak{gl}(V)$ move down the poset and these moves coincide with jumps in border ranks. In Figure 2 we record the dominance poset for partitions of 6 together with the border ranks of the monomials with the exponent vector equal to the given partition, and we labeled some lowering operator moves on the poset. This idea led to the concept of partial Young flattenings. Our implementation of partial Young flattenings is included as an ancillary file to the arXiv version of this article. We include an example of the partial Young flattening construction for $x_0^2x_1^2x_2^2x_3$. For this we constructed 18 separate maps, each of which took between 20s and 700s to construct. It then took less than a second to verify that each of the summands has rank 15, which gives the lower bound of 18 on the border rank.

5. A (partial) $\mathfrak{gl}(V_0)$-action on Young flattenings.

After many examples were computed we noticed that Young flattenings of monomials are block diagonal (after some permuting of rows and columns). So it is natural to try to determine the blocking structure and to compute the rank of each block.

One first guess might be to decompose the source and target of the map via the branching $\mathfrak{gl}(n+1) \downarrow \mathfrak{gl}(n)$, however this turns out not to be the right tool because a general Young flattening won't exactly be $\mathfrak{gl}(n)$-equivariant for $\mathfrak{gl}(n) \subset \mathfrak{gl}(n+1)$. Another first guess might be to decompose source and target as torus modules, however this method turns out not to be so useful as many Young flattenings are not bijective or full-rank on weight spaces.

Our method is to use the action of $\mathfrak{gl}(V)$ on Young flattenings in two parts. First we use elements in $H_0 \times \mathrm{Hom}(\langle x_0 \rangle, V_0) \subset \mathfrak{gl}(V)$ to decompose the Young flattening into a direct sum of maps. While the Young flattening itself may not be $\mathfrak{gl}(V_0)$-equivariant, each summand has a separate $\mathfrak{gl}(V_0)$ action, is equivariant for this action, and Shur's lemma allows us to easily compute the rank of each summand.

Notation: For $\alpha = (\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n) \in \mathbb{N}^{n+1}$ with $d = \sum_{i=0}^n \alpha_i$. Let $\alpha' = (\alpha_1, \ldots, \alpha_n)$. If $\mu, \nu \in \mathbb{N}^n$ we will write $\mu \preceq \alpha'$ if we have a partition $\alpha' = \mu + \nu$. Recall that a partition λ is α-optimal if $\lambda = (\sum_{i=1}^n \alpha_i, \sum_{i=1}^{n-1} \alpha_i, \ldots, \alpha_1)$. Also recall our notation $X_i^j = \boxed{j} \cdot \frac{\partial}{\partial \boxed{i}}$ acting on tableaux, and $X_0^\gamma := (X_0^n)^{\gamma_n} (X_0^{n-1})^{\gamma_{n-1}} \cdots (X_0^1)^{\gamma_1}$, and $H_0(T) = \mathcal{F}_\lambda(x_0^d)(T) \lrcorner 0^d$, for each λ with $\lambda_1 \leq d$.

Definition 5.1. We define two distinguished vector spaces, $\mathcal{A}(\mu)$ and $\mathcal{B}(\nu)$, adapted respectively for the source and target of a (partial) Young flattening.

Set $\mathcal{K}_\lambda(\mu) := \ker H_0 X_0^\mu \colon S_\lambda V \to S_\lambda V$. Define $\mathcal{A}_\lambda(\mu)$ by choosing a complement to $\mathcal{K}_\lambda(\mu)$ giving the vector space decomposition

$$S_\lambda V = \mathcal{A}_\lambda(\mu) \oplus \mathcal{K}_\lambda(\mu).$$

In particular $\mathcal{A}(\mu) \cong S_\lambda V / \mathcal{K}_\lambda(\mu) \cong H_0 X_0^\mu S_\lambda V$ (by the first isomorphism theorem). We usually drop λ from the notation when it is understood.

For each partition ν define $\mathcal{B}_\lambda(\nu)$ (or $\mathcal{B}(\nu)$ when λ is understood) and $\mathcal{B}^p(\nu)$ as follows:

$$\mathcal{B}(\nu) := X_0^\nu \mathcal{F}_\lambda(x_0^d)(S_\lambda V_0), \quad \text{and} \quad \mathcal{B}^p(\nu) := X_0^\nu \otimes \mathcal{F}_\lambda(x_0^d)(S_\lambda V_0).$$

We note that $\mathcal{B}^p(\nu)$ are the unevaluated cousins of $\mathcal{B}(\nu)$ and the $\mathcal{B}^p(\nu)$ are isomorphic to $S_\lambda V_0$ as vector spaces.

5.1. $\mathfrak{gl}(V_0)$-module structure.

Lemma 5.4 says that when λ is α-optimal $\mathcal{A}(\mu)$ is isomorphic to $S_\lambda V_0$, and this isomorphism of vector spaces can be used to give $\mathcal{A}(\mu)$ the structure of a $\mathfrak{gl}(V_0)$-module. Specifically, if $T \in \mathcal{A}(\mu)$, and $g \in \mathfrak{gl}(V_0)$ define $g.T$ to be the element $Y \in \mathcal{A}(\mu)$ such that

$$g.H_0 X_0^\mu(T) = H_0 X_0^\mu Y,$$

with $H_0 X_0^\mu(T) \in S_\lambda V_0$ and $g.H_0 X_0^\mu(T)$ the usual $\mathfrak{gl}(V_0)$ action on $S_\lambda V_0$. The existence and uniqueness of such a Y comes from the fact that $H_0 X_0^\mu \colon A(\mu) \to S_\lambda V_0$ is an isomorphism.

However, we note that this $\mathfrak{gl}(V_0)$-action will not be the same as the action of the copy of $\mathfrak{gl}(V_0)$ in $\mathfrak{gl}(V)$, otherwise $\mathcal{A}(\mu)$ can fail to be closed: See Example 4.6, where the standard $\mathfrak{gl}(V_0)$ acting on $\mathcal{A}(1)$ would intersect $\mathcal{A}(2)$, whereas these modules are independent copies of $S_{2,1}\mathbb{C}^2$, and closed under the action we define.

We can give $\mathcal{B}(\nu)$ the structure of a $\mathfrak{gl}(V_0)$-module as follows. For each $g \in \mathfrak{gl}(V_0)$ and for each $T \in S_\lambda V_0$, we set

$$g.\left(X_0^\nu F(x_0^d)(T)\right) = X_0^\nu F(x_0^d)(g.T), \tag{10}$$

with $g.T$ being the usual $\mathfrak{gl}(V_0)$-action on $S_\lambda V_0$. It is clear that $\mathcal{B}(\nu)$ is closed under this action of $\mathfrak{gl}(V_0)$. The action of $\mathfrak{gl}(V_0$ on $\mathcal{B}^p(\nu)$ is similarly defined.

If we were to view $\mathfrak{gl}(V_0)$ inside of $\mathfrak{gl}(V)$, we would define the action as:

$$g.\left(X_0^\nu F(x_0^d)(T)\right) = (g.X_0^\nu)F(x_0^d)(T) + X_0^\nu F(x_0^d)(g.T),$$

however, $\mathcal{B}(\nu)$ will not be closed under this action in general.

Definition 5.2. A vector $v \in \mathcal{A}(\mu)$ is called a *highest weight vector*, *hwv* for short, if considering the action of $\mathfrak{gl}(V_0)$ on $\mathcal{A}(\mu)$, v is in the kernel of all elements of the subspace of raising operators $\mathfrak{n}_+ \subset \mathfrak{gl}(V_0)$. Lowest weight vectors in $\mathcal{A}(\mu)$, *lwv*'s for short, are similarly defined.

Remark 5.3. We note that if v is a hwv in $\mathcal{A}(\mu)$ it will have weight $\alpha' + \mu$, the weight of a highest weight vector in $S_\lambda V$ shifted by the action of $H_0 X_0^\mu$. An example is $\mu = (1)$, and $S_\lambda V = S_d V$, where in this case $\mathcal{A}(1) \ncong S_d V_0$.

Now we can give a proof of Theorem 1.4, which we restate here for convenience.

Theorem 1.4. *Let $\alpha, \beta \in \mathbb{N}^{n+1}$ with $|\alpha| = |\beta_i| = d$. Suppose λ is an α-optimal partition.*
1. *If $\beta' \preceq \alpha'$, the partial Young flattening $\mathcal{F}_\lambda^p(x^\beta)$ has rank $\dim S_\lambda V_0 \cdot \prod_{i=1}^n (\beta_i + 1)$.*
2. *If $\beta' \preceq \alpha'$ and $(\beta_1, \ldots, \beta_n) \preceq (\alpha_0, \alpha_n, \alpha_{n-1}, \ldots, \alpha_2)$, then the Young flattening $\mathcal{F}_\lambda(x^\beta)$ has rank $\dim S_\lambda V_0 \cdot \prod_{i=1}^n (\beta_i + 1)$.*

Proof of Theorem 1.4. We will show that the (partial) flattening $\mathcal{F}_\lambda(x^\alpha)$ (respectively $\mathcal{F}_\lambda^p(x^\alpha)$) is a direct sum of $\mathfrak{gl}(V_0)$-equivariant maps, each of which is a copy of the map $\mathcal{F}_\lambda(x_0^d)$, and the number of summands will be determined by the shape of λ (the best situation from the point of view of border ranks of monomials is when λ is α-optimal).

We can move between (scalar multiples of) x^α and x_0^d via lowering and raising operators:

$$[x_0^{n+1}] = [(X_n^0)^{\alpha_n} \ldots (X_2^0)^{\alpha_2}.(X_1^0)^{\alpha_1}.(x^\alpha)], \quad \text{and} \quad [x^\alpha] = [(X_0^n)^{\alpha_n} \ldots (X_0^2)^{\alpha_2}.(X_0^1)^{\alpha_1}.(x_0^d)].$$

In particular

$$[\mathcal{F}_\lambda(X_0^\alpha.(x_0^d))] = [\mathcal{F}_\lambda(x^\alpha)] \quad \text{and} \quad [\mathcal{F}_\lambda^p(X_0^\alpha.(x_0^d))] = [\mathcal{F}_\lambda^p(x^\alpha)].$$

Iteratively apply Proposition 4.2 and its analogue in the partial Young flattening case to obtain (analogous to (7) and (8) above) for $T \in S_\lambda V$

$$\mathcal{F}\left(X_0^{\alpha'}.(x_0^d)\right)(T) = \sum_{\mu+\nu=\alpha'} (-1)^{|\mu|} X^\nu.\mathcal{F}(x_0^d)(X^\mu.T), \tag{11}$$

$$\mathcal{F}^p\left(X_0^{\alpha'}.(x_0^d)\right)(T) = \sum_{\mu+\nu=\alpha'} (-1)^{|\mu|} \mathcal{F}(x_0^d)(X^\mu.T) \otimes X^\nu. \tag{12}$$

The source of both maps is $\sum_{\mu \preceq \alpha'} \mathcal{A}(\mu)$, which is a direct sum by Lemma 5.5. Each $\mathcal{A}(\mu) \cong S_\lambda V_0$ by Lemma 5.4. The target of the partial Young flattening (12) is $\sum_{\nu \preceq \alpha'} \mathcal{B}^p(\nu)$. Each term $\mathcal{B}^p(\nu) = S_\lambda V_0 \otimes \langle x_0 \rangle \otimes \langle X_0^\nu \rangle$ is isomorphic to $S_\lambda V_0$ by construction and the set $\{X_0^\nu \mid \nu \preceq \alpha'\}$ is linearly independent, so the sum $\sum_{\nu \preceq \alpha'} \mathcal{B}^p(\nu)$ is direct.

The number of summands both (11) and (12), is the number of distinct partitions $\mu + \nu = \alpha'$, which is also equal to the number of polynomials in variables $x_1, \ldots, x_n$ of multi-degree

at most α', or $\prod_{i=1}^{n}(\alpha_i+1)$. This proves Part (1). Part (2) follows from the additional hypothesis since in this case Lemma 5.10 implies that the contractions $\mathcal{B}^p(\nu)\to\mathcal{B}(\nu)$ are all isomorphisms and the $\mathcal{B}(\nu)$ are linearly independent. Example 5.9 shows the smallest example where Young flattenings don't give the best bound on border rank, and thus the first example where partial Young flattenings are needed. So the theorem is proved pending the technical lemmas. ☐

Lemma 5.4. *Suppose λ is α-optimal and $\mu\preceq\alpha'$. Then $H_0X_0^{\mu}(S_\lambda V)=H_0X_0^{\mu}(\mathcal{A}(\mu))=S_\lambda V_0$. In particular, $\mathcal{A}(\mu)\cong S_\lambda V_0$.*

Proof. The equality $H_0X_0^{\mu}(S_\lambda V)=H_0X_0^{\mu}(\mathcal{A}(\mu))$ holds by the definition of $\mathcal{A}(\mu)$. Since $H_0S_\lambda V=S_\lambda V_0$, we can conclude by showing that the image of $H_0X_0^{\mu}$ contains every basis vector of $S_\lambda V_0$. Suppose T is a tableau in $\mathrm{SSYT}_\lambda\{1,\dots,n\}$. Since λ is α-optimal Lemma 2.4 guarantees that T has a GHS with content μ. Select one such GHS and replace it with $|\mu|$ $\boxed{0}$'s. The resulting tableau might not be semi-standard but is a *non-zero* element of $S_\lambda V$ (replacing a GHS with a letter higher than all letters in the given tableau reduce to zero modulo shuffling relations) and maps to a non-zero point on the line $[T]$ via $H_0X_0^{\mu}$. ☐

Lemma 5.5. *Suppose λ is α-optimal and $\mu,\nu\preceq\alpha'$. Then $\mathcal{A}(\mu)$ and $\mathcal{A}(\nu)$ are either linearly independent (when $\mu\neq\nu$) or they coincide.*

Proof. Note that Lemma 5.4 implies that $\mathcal{A}(\mu)\cong\mathcal{A}(\nu)\cong S_\lambda V_0$. We show that the highest weight vector of one cannot be an element of the other, using the GHS finding lemma.

Suppose $0\neq T\in\mathcal{A}(\mu)\cap\mathcal{A}(\nu)$ is a vector of highest possible weight in the intersection, and $0\neq H_0X_0^{\mu}T\in S_\lambda V_0$. If $H_0X_0^{\mu}T$ is not on the highest weight line of $S_\lambda V_0$, apply $g\in\mathfrak{n}_+\subset\mathfrak{gl}(V_0)$. Since $g.H_0X_0^{\mu}T$ is nonzero in $S_\lambda V_0$, by Lemma 5.4 there must be a $Y\in\mathcal{A}_\mu$ and a $Y'\in\mathcal{A}(\nu)$ such that

$$H_0X_0^{\mu}Y=gH_0X_0^{\mu}T\quad\text{and}\quad H_0X_0^{\nu}Y'=gH_0X_0^{\mu}T.$$

The action of $\mathfrak{gl}(V_0)$ on $\mathcal{A}(\mu)$ is defined so that $gH_0X_0^{\mu}T=H_0X_0^{\mu}(g.T)$. Thus

$$H_0X_0^{\nu}Y'=gH_0X_0^{\mu}T=H_0X_0^{\mu}(g.T).$$

Therefore $0\neq g.T\in\mathcal{A}(\mu)\cap\mathcal{A}(\nu)$ for every nonzero T of non-maximal weight. So the intersection must contain the highest weight vector of each of $\mathcal{A}(\mu)$ and $\mathcal{A}(\nu)$. A similar argument applies to lowest weight vectors.

So $\mathcal{A}(\mu)\cap\mathcal{A}(\nu)$ contains the respective highest and lowest weight vectors of each, and by applying the respective actions of $\mathfrak{gl}(V_0)$ the intersection must contain both copies of $S_\lambda V_0$. This is a contradiction unless the two copies coincide since $\mathcal{A}(\mu)\cong\mathcal{A}(\nu)\cong S_\lambda V_0$, and their intersection cannot be larger. Finally, since $\mu\neq\nu$, the respective hwv's have different weights in $S_\lambda V$, so the two copies cannot coincide, and the intersection must be zero. ☐

Remark 5.6. We would like to say that this follows from the Schur's lemma since both are irreducible, however, they have different copies of $\mathfrak{gl}(V_0)$ acting on them, so inclusion is not an equivariant map. Moreover, we have counter-examples when the shape is not optimal: shape (3,1) going to shape (4,3,1). In this case $\mathcal{A}_{3,1}(1,1)=\left\langle \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 0&0&2\\\hline 2\\\cline{1-1}\end{array}\right\rangle$, which overlaps $\mathcal{A}_{3,1}(1,2)=\left\langle \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 1\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 0&0&2\\\hline 1\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 0&0&2\\\hline 2\\\cline{1-1}\end{array}\right\rangle$, and the two don't even have the same dimension. Moreover, neither of these vector spaces are 3-dimensional, the dimension of $S_{3,1}\mathbb{C}^2$. This illustrates the kinds of problems one encounters when non-optimal shapes are used.

Lemma 5.7. *If λ is α-optimal and $\mu \neq \nu \preceq \alpha'$, then $\mathcal{A}(\mu) \cap \mathcal{A}(\nu) = \{0\}$.*

Proof. If $\mu \neq \nu \preceq \alpha'$ then the hwv's of $\mathcal{A}(\mu)$ and $\mathcal{A}(\nu)$ have different weights, so one must not contain the other by Lemma 5.5. $\square$

Lemma 5.8. *If λ is α-optimal and $\mu \neq \nu \preceq \alpha'$ then $H_0 X_0^\nu \mathcal{A}(\mu) = 0$.*

Proof. Lemma 5.4 says that $H_0 X_0^\mu (S_\lambda V) = H_0 X_0^\mu (\mathcal{A}(\mu)) = S_\lambda V_0$. Lemma 5.5 says that $\mathcal{A}(\mu)$ and $\mathcal{A}(\nu)$ are independent, which means that $\mathcal{A}(\nu)$ is in the kernel of $H_0 X_0^\mu$. $\square$

Monomial-optimal flattenings can fail to have a large enough target, the smallest example of which is the following.

Example 5.9. Consider $\lambda = (5, 4, 2)$, which has weight $(0, 1, 2, 2)$ and is the optimal shape for $\varphi = x_0^2 x_1^2 x_2^2 x_3$. The map $\mathcal{F}_{5,4,2}(\varphi)$ is size 300×360, and has rank 255 for φ, versus rank 15 for x_0^7. This means that the lower bound we get is only 17, which is less than the predicted 18. It is still true that the source splits as a direct sum $\bigoplus_{\mu \preceq \alpha'} X_0^\mu S_{5,4,2} V_0$. However, the targets $X_0^\nu \mathcal{F}_{5,4,2}(x_0^7) S_{5,4,2} V_0$ are not all isomorphic to $S_{5,4,2} V_0$ – some are of smaller dimension. The problem is that the contragradient partition to λ in a 4×7 box is $(7, 5, 3, 2)$, which has weight $(2, 2, 1, 2)$ and is not monomial-optimal for $\alpha = (2, 2, 2, 1)$, and there can be tableaux that don't support the GHS $(2, 2, 2, 1)$. More specifically, the tableau $\begin{ytableau} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 \end{ytableau}$ can't have a GHS with content 0011223 added with no repeats in columns. Note also that the contragradient tableau $\begin{ytableau} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 \\ 3 & 3 \end{ytableau}$ does not have a GHS with content 0011223.

On the other hand, the highest weight tableau $\begin{ytableau} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 \end{ytableau}$ can have a GHS with content 00011223 added with no repeats in columns: $\begin{ytableau} 0 & 0 & 0 & 3 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 \end{ytableau}$, and so can all other SSYT of shape (5,4,2). The contragradient tableau of shape $(8, 6, 4, 3)$ is $\begin{ytableau} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 \end{ytableau}$ which does have a GHS with content 00011223. The lowest weight tableau $\begin{ytableau} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 3 \end{ytableau}$ can have a GHS with content 00011223 added with no repeats in columns: $\begin{ytableau} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 0 \\ 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 3 \end{ytableau}$. Also, the contragradient tableau of shape $(8, 6, 4, 3)$ is $\begin{ytableau} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 \end{ytableau}$ which does have a GHS with content 00011223. These checks imply that the flattening $F_{5,4,2}(x_0^3 x_1^2 x_2^2 x_3)$ has rank $270 = 18 \cdot 15$, which implies that $x_0^3 x_1^2 x_2^2 x_3$ has border rank 18.

The smallest degree example where this behavior occurs is 7. We have verified that the optimal shape for each monomial of degree ≤ 6 has contragradient that dominates the shape of the monomial. See Table 1 for the sizes and ranks of the flattenings in the case $d = 7$. On the other hand, we were able to verify that $x_0 \varphi = x_0^3 x_1^2 x_2^2 x_3$ has border rank 18 by checking the rank of the flatting $\mathcal{F}_{8,6,4,2}(x_0^3 x_1^2 x_2^2 x_3)$ is 486, whereas $\mathcal{F}_{8,6,4,2}(x_0^8)$ has rank 27. $\diamondsuit$

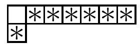
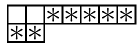
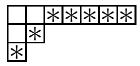
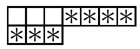
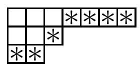
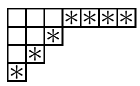
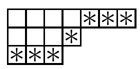
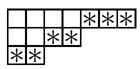
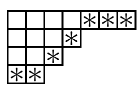
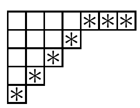
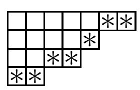
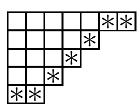
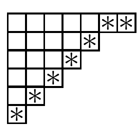
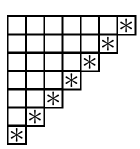
φ	flattening	size	Rank $\mathcal{F}(x_0^7)$	Rank $\mathcal{F}(\varphi)$	Rank $\mathcal{F}^p(\varphi)$
$x_0^6x_1$		7×2	1	2	2
$x_0^5x_1^2$		6×3	1	3	3
$x_0^5x_1x_2$		48×8	2	8	8
$x_0^4x_1^3$		5×4	1	4	4
$x_0^4x_1^2x_2$		35×15	2	12	12
$x_0^4x_1x_2x_3$		420×64	8	64	64
$x_0^3x_1^3x_2$		24×24	2	16	16
$x_0^3x_1^2x_2^2$		42×27	3	27	27
$x_0^3x_1^2x_2x_3$		256×140	8	96	96
$x_0^3x_1x_2x_3x_4$		5120×1024	64	1024	1024
$x_0^2x_1^2x_2^2x_3$		360×300	15	255	270
$x_0^2x_1^2x_2x_3x_4$		2520×2520	64	1536	1536
$x_0^2x_1x_2x_3x_4x_5x_6$		88704×2^{15}	2^{10}	2^{15}	2^{15}
$x_0x_1x_2x_3x_4x_5x_6x_7$		$2^{21} \times 2^{21}$	2^{15}	2^{21}	2^{21}

Table 1. Optimal (partial) Young flattenings for all monomials of degree 7

We can overcome the issue seen in Example 5.9 by working with partial Young flattenings. The following gives the criteria to avoid this issue for usual Young flattenings.

Lemma 5.10. *Suppose λ^* is α-optimal, and $d \geq \lambda_1$. For each $\nu \preceq \alpha'$ define the vector spaces $\mathcal{B}(\nu) := X_0^{\nu}\mathcal{F}_\lambda(x_0^d)(S_\lambda V_0)$. Then*

1. *Each $\mathcal{B}(\nu)$ is isomorphic to $S_\lambda V_0$.*
2. *The $\mathcal{B}(\nu)$ are mutually linearly independent in $S_{d,\lambda}V$.*

φ	flattening	size	Rank $\mathcal{F}(x_0^6)$	Rank $\mathcal{F}(\varphi)$
$x_0^5x_1$	$\ytableaushort{{}{\ast}{\ast}{\ast}{\ast}{\ast},{\ast}}$	2×6	1	2
$x_0^4x_1^2$	$\ytableaushort{{}{}{\ast}{\ast}{\ast}{\ast},{\ast}{\ast}}$	3×5	1	3
$x_0^3x_1^3$	$\ytableaushort{{}{}{}{\ast}{\ast}{\ast},{\ast}{\ast}{\ast}}$	4×4	1	4
$x_0^4x_1x_2$	$\ytableaushort{{}{}{\ast}{\ast}{\ast}{\ast},{}{\ast},{\ast}}$	8×35	2	8
$x_0^3x_1^2x_2$	$\ytableaushort{{}{}{}{\ast}{\ast}{\ast},{}{}{\ast},{\ast}{\ast}}$	15×24	2	12
$x_0^2x_1^2x_2^2$	$\ytableaushort{{}{}{}{}{\ast}{\ast},{}{}{\ast}{\ast},{\ast}{\ast}}$	27×27	3	27
$x_0^3x_1x_2x_3$	$\ytableaushort{{}{}{}{\ast}{\ast}{\ast},{}{}{\ast},{}{\ast},{\ast}}$	64×256	8	64
$x_0^2x_1^2x_2x_3$	$\ytableaushort{{}{}{}{}{\ast}{\ast},{}{}{}{\ast},{}{}{\ast},{\ast}{\ast}}$	140×140	8	96
$x_0^2x_1x_2x_3x_4$	$\ytableaushort{{}{}{}{}{\ast}{\ast},{}{}{}{\ast},{}{}{\ast},{}{\ast},{\ast}}$	1024×2560	64	1024
$x_0x_1x_2x_3x_4x_5$	$\ytableaushort{{}{}{}{}{}{\ast},{}{}{}{}{\ast},{}{}{}{\ast},{}{}{\ast},{}{\ast},{\ast}}$	$2^{15} \times 2^{15}$	2^{10}	2^{15}

TABLE 2. Optimal Young flattenings for all monomials of degree 6

Proof. For part (1), note that if the statement is true for $\nu = \alpha'$ then it is true for all $\nu \preceq \alpha'$. So we proceed with $\nu = \alpha'$. Suppose T is a tableau in $\mathrm{SSYT}_\lambda\{1, \dots, n\}$. Lemma 2.6 says that adding a GHS with content $(0^{d-|\alpha'|}, \alpha') = \alpha$ to the top of T produces a non-zero linear combination of tableaux in $S_{d,\lambda}V$. Then the rest of the proof is the same as in Lemma 5.4, which concludes the proof of (1). For part (2), notice that the statement is true if and only if the contragradients $\mathcal{B}(\nu)^*$ are linearly independent in the contragradient $S_{d,\lambda}V^* = S_{(d,\lambda)^*}V$, which follows from Lemma 5.7. □

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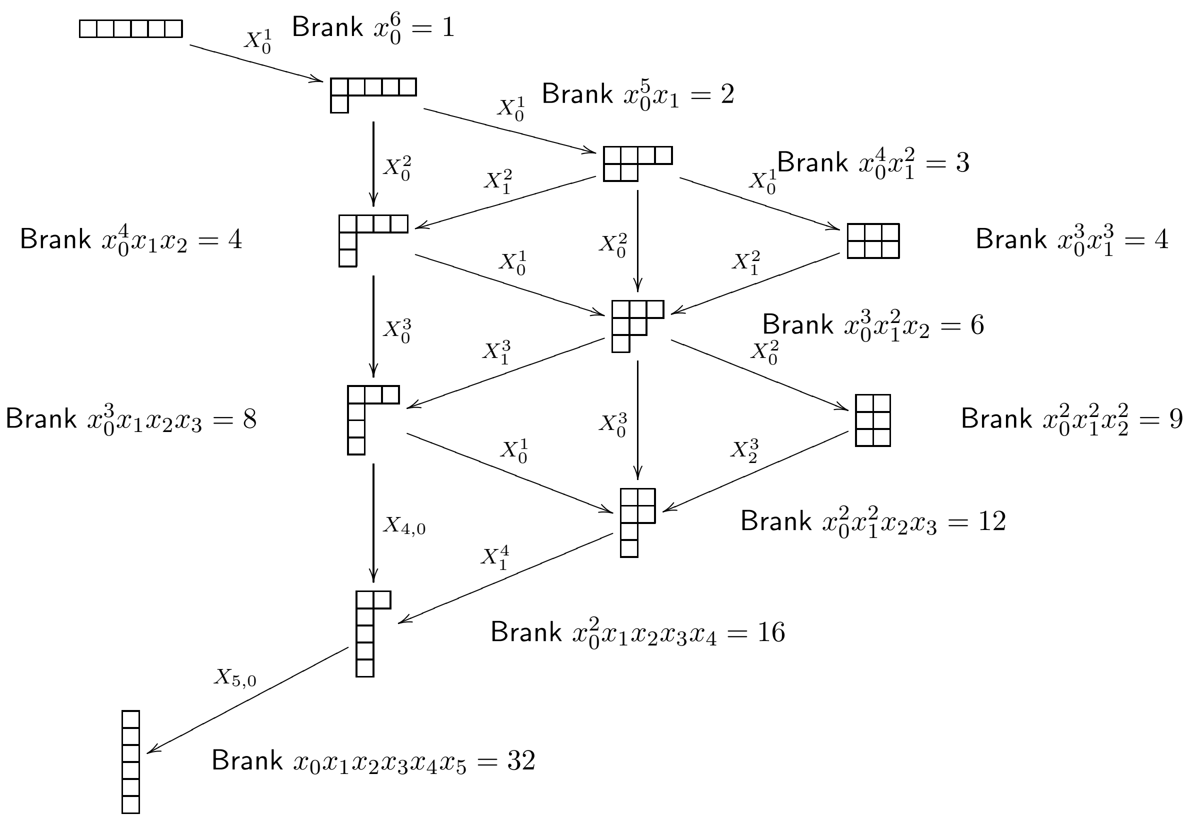

FIGURE 2. The dominance poset for sextic monomials and their border ranks

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